The Kostant form of $\mathfrak{U}(sl_n^+)$ and the Borel subalgebra of the Schur algebra $S(n,r)$

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Abstract

Let $\mathfrak{A}_n(\mathbb{K})$ be the Kostant form of $\mathfrak{U}(sl_n^+)$ and Γ the monoid generated by the positive roots of sl_n. For each $\lambda \in \Lambda(n,r)$ we construct a functor F_λ from the category of finitely generated Γ-graded $\mathfrak{A}_n(\mathbb{K})$-modules to the category of finite-dimensional $S^+(n,r)$-modules, with the property that F_λ maps (minimal) projective resolutions of the one-dimensional $\mathfrak{A}_n(\mathbb{K})$-module $\mathbb{K}_{\mathfrak{A}}$ to (minimal) projective resolutions of the simple $S^+(n,r)$-module $\mathbb{K}_\lambda$.

Introduction

The polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$ were studied by I. Schur in his doctoral dissertation [16]. In this famous work, Schur introduced the, now called, Schur algebras, which are a powerful tool to connect r-homogeneous polynomial representations of the symmetric group on r symbols.

These results of I. Schur were generalised by J.-A. Green to infinite fields of arbitrary characteristic in [10]. In Green's work the Schur algebra $S(n,r) = S_{\mathbb{K}}(n,r)$ plays the central role in the study of polynomial representations of $\mathrm{GL}_n(\mathbb{K})$.

In [7] Donkin shows that $S(n,r)$ is a quasi-hereditary and so it has finite global dimension. This led to the problem of describing explicit projective resolution of the Weyl modules for $S(n,r)$. Only partial answers to this problem are known. In [1] and [19] such resolutions were constructed for the case when $\mathbb{K}$ is a field of characteristic zero. If $\mathbb{K}$ has arbitrary characteristic then projective resolutions of W_λ are given in [2] when $n = 2$ (λ arbitrary), and in [13] and [17] for hook partitions.

In [17] Woodcock provides the tools to reduce the problem of constructing these resolutions to the similar problem for the simple modules for the Borel subalgebra $S^+(n,r)$ of $S(n,r)$.

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Denote by $\Lambda(n,r)$ the set of compositions of r onto n parts. It is proved in [15] that all simple $S^+(n,r)$-modules are one-dimensional and parametrised by the set $\Lambda(n,r)$. We denote the simple module corresponding to $\lambda \in \Lambda(n,r)$ module by $\mathbb{K}_\lambda$. In [15], the first two steps in a minimal projective resolution of $\mathbb{K}_\lambda$ and the first three terms of a minimal projective resolution in the case $n = 2$ are constructed. In [18] minimal projective resolutions for $\mathbb{K}_\lambda$ for $\lambda \in \Lambda(2,r)$ and non-minimal projective resolutions of $\mathbb{K}_\lambda$ for $\lambda \in \Lambda(3,r)$ are constructed. The results of both papers depend on heavy calculations in the algebra $S^+(n,r)$.

In the present paper we take a more abstract approach.

Let us denote by $\mathfrak{A}_n(\mathbb{K})$ the Kostant form over the field $\mathbb{K}$ of the universal enveloping algebra of the Lie algebra sl_n^+ of upper triangular nilpotent matrices. Then $\mathfrak{A}_n(\mathbb{K})$ has a unique one-dimensional module, which we denote by $\mathbb{K}_{\mathfrak{A}}$. In this paper we show that the construction of (minimal) projective resolutions for K_λ is essentially equivalent to the construction of (minimal) projective resolution for $\mathbb{K}_{\mathfrak{A}}$. The last task is much more feasible, since an explicit presentation of $\mathfrak{A}_n(\mathbb{K})$ can be given and thus the results of Anick [3] can be applied to the description of an explicit projective resolution of $\mathbb{K}_{\mathfrak{A}}$. It is also worth to note that $\mathfrak{A}_n(\mathbb{K})$ is a projective limit of finite dimensional algebras in the case $\mathrm{char}(\mathbb{K}) = p > 0$, and therefore the technique developed in [5] can be used for the construction of the minimal projective resolution of $\mathbb{K}_{\mathfrak{A}}$. This line of research will be followed by us in the subsequent papers.

The general plan of the present paper is as follows. In Section 1 we collect general technical results, which will be used in the following section. We believe that these results can be applied in more general context, in particular to the generalised and q-Schur algebras.

Let G be an ordered group with neutral element ϵ. Denote by $\Gamma \subset G$ the submonoid of non-negative elements of G. For every Γ-graded algebra A and every Γ-set S, we construct a family of Γ-algebras

$$\{C(X) | X \subset S\}$$

and a family of Γ-graded algebra homomorphisms

$$\left\{\phi_X^Y \colon C(Y) \to C(X) \middle| X \subset Y \subset S\right\}$$

such that $\phi_X^Y \circ \phi_Y^Z = \phi_X^Z$ for every triple $X \subset Y \subset Z$ of subsets in S. In other words, $C(-)$ is a presheaf of Γ-graded algebras.

For every $x \in S$, we construct an exact functor F_x from the category $\mathscr{C}(A,\Gamma)$ of finitely generated Γ-graded A-modules to the category $\mathscr{C}(C(S),\Gamma)$. If Γ acts by automorphisms on S, then the functors F_x preserve projective modules, and thus map projective resolutions into projective resolutions. If $A_\epsilon \cong \mathbb{K}$, then the functors F_x map minimal projective resolutions into minimal projective resolutions.

For every $X \subset S$, we can consider each left $C(X)$-module as a left $C(S)$-module via a homomorphism $\phi_X^S \colon C(S) \to C(X)$. Thus we get a natural inclusion of categories

$$(\phi_X^S)^* \colon \mathscr{C}(C(X),\Gamma) \to \mathscr{C}(C(S),\Gamma).$$

There is a left adjoint functor to $\left(\phi_X^S\right)^*$

$$(\phi_X^S)_* = C(X) \otimes_{C(S)} -: \mathscr{C}(C(S), \Gamma) \to \mathscr{C}(C(X), \Gamma),$$

where we consider $C(X)$ as a right $C(S)$-module via ϕ_X^S. The main objective of Section 1.4 is to get conditions on $X \subset S$, ensuring that for every left $C(X)$-module M and every (minimal) projective resolution $P_\bullet$ of $\left(\phi_X^S\right)^*(M)$ the complex $\left(\phi_X^S\right)_*(P_\bullet)$ is a (minimal) projective resolution of $M \cong \left(\phi_X^S\right)_* \left(\phi_X^S\right)^*(M)$. If the both algebras $C(S)$ and $C(X)$ are artinian and $\left(\phi_X^S\right)_*$ has the above mentioned property, then the ideal $\mathrm{Ker}\left(\phi_X^S\right)$ is a *strong idempotent ideal*. The algebra $C(X)$ is finite dimensional and thus artinian, if X is finite and A is locally finite dimensional. But the algebra $C(S)$ is rarely finite dimensional. To cope with this, we take a two stage approach.

We say that $Y \subset S$ is Γ-convex, if from $\gamma = \gamma_1\gamma_2$ and x, $\gamma x \in Y$ it follows that $\gamma_2 x \in Y$. In Proposition 1.29, we show that if Y is a convex Γ-set then the functor $\left(\phi_Y^S\right)_*$ is exact and maps (minimal) projective resolutions into (minimal) projective resolutions.

Let Y be a finite Γ-convex subset of S and X a subset of Y. Suppose that A is locally finite dimensional. In Theorem 1.35 we give a criterion for $\mathrm{Ker}\left(\phi_X^Y\right)$ to be a strong idempotent ideal.

In Section 2 we apply the results of Section 1 to $S^+(n,r)$. In our particular case, the algebra A is the Kostant form $\mathfrak{A}_n(\mathbb{K})$, the set S is $\mathbb{Z}^n$ and Γ is the submonoid of $\mathbb{Z}^n$ generated by the elements $(0,\dots,1,-1,\dots,0)$. Then we show in Theorem 2.20 that $C(\Lambda(n,r)) \cong S^+(n,r)$. Here we consider $\Lambda(n,r)$ as a subset $\mathbb{Z}^n$ in the natural way. Note that this isomorphism gives a description of $S^+(n,r)$, which is similar to the idempotent presentation of the algebra $S(n,r)$ obtained by Doty and Giaquinto in [8].

The set of compositions $\Lambda(n,r)$ is contained in the lager finite set $\Lambda^1(n,r)$ defined by

$$\Lambda^1(n,r) = \left\{ (z_1, z_2, \dots, z_n) \in \mathbb{Z}^n \,\middle|\, 0 \le \sum_{i=1}^{j} z_i,\ 1 \le j \le r-1; \sum_{i=1}^{n} z_i = r \right\}.$$

It turns out, that $\Lambda^1(n,r)$ is a Γ-convex set (see Proposition 2.4) and therefore the functor $\left(\phi^{\mathbb{Z}^n}_{\Lambda^1(n,r)}\right)_*$ is exact and preserves (minimal) projective resolutions. Moreover, we show in Theorem 2.19 that $\mathrm{Ker}\left(\phi^{\Lambda^1(n,r)}_{\Lambda(n,r)}\right)$ is a strong idempotent ideal. Hence the composite functor

$$\left(\phi^{\Lambda^1(n,r)}_{\Lambda(n,r)})\right)_* \circ \left(\phi^{\mathbb{Z}^n}_{\Lambda^1(n,r)}\right)_* = \left(\phi^{\mathbb{Z}^n}_{\Lambda(n,r)}\right)_*$$

preserves projective resolutions of $C(\Lambda(n,r)) = S^+(n,r)$-modules considered as $C(\mathbb{Z}^n)$-modules.

Recall, that $\mathbb{K}_{\mathfrak{A}}$ denotes the unique one-dimensional module over $\mathfrak{A}_n(\mathbb{K})$ and F_λ is the functor from $\mathscr{C}(\mathfrak{A}_n(\mathbb{K}), \Gamma)$ to $\mathscr{C}(C(\mathbb{Z}^n), \Gamma)$ associated with $\lambda \in \mathbb{Z}^n$. It

follows from the definitions, that

$$F_\lambda(\mathbb{K}_{\mathfrak{A}}) \cong \left(\phi^{\mathbb{Z}^n}_{\Lambda(n,r)}\right)^* (\mathbb{K}_\lambda).$$

Therefore, if $P_\bullet$ is a (minimal) projective resolution of $\mathbb{K}_{\mathfrak{A}}$, then

$$\left(\phi^{\mathbb{Z}^n}_{\Lambda(n,r)}\right)_* \circ F_\lambda(P_\bullet)$$

is a (minimal) projective resolution of $\mathbb{K}_\lambda$.

Now we give a more detailed outline of the paper. In Section 1.1, we make a (fairly simple) extension of Eilenberg machinery on perfect categories to the case of Γ-graded algebras. Most results are valid only for positive monoids, that is for submonoids of non-negative elements in an ordered group.

In Section 1.2 we define the skew product $A \ltimes_\Gamma B$ of a Γ-graded algebra A and Γ-algebra B over the monoid Γ. If Γ is a group G and A is the group algebra $\mathbb{K}[G]$, then $\mathbb{K}[G] \ltimes_G B$ is isomorphic to the usual skew product of B with G. For every B-module N we construct an exact functor

$$- \ltimes_\Gamma N \colon \mathscr{C}(A,\Gamma) \to \mathscr{C}(A \ltimes_\Gamma B, \Gamma),$$

and establish conditions for $- \ltimes_\Gamma N$ to preserve (minimal) projective resolutions.

Section 1.3 is an overview of results of homological algebra, which we use after. In particular, we recall the notions of strong idempotent ideal and of heredity ideal and some of their properties.

Section 1.4 is the central section of the first part of our work. Here we prove a criterion for $\operatorname{Ker}\left(\phi^Y_X\right)$ to be a strong idempotent ideal.

In Section 2.1 we prove the results about compositions, multi-indices and orderings on $\mathbb{Z}^n$, which we use later on.

The Schur algebra $S(n,r)$ and its Borel subalgebra $S^+(n,r)$ as well as the algebra $\mathfrak{A}_n(\mathbb{K})$ are considered in Sections 2.2 and 2.3, respectively.

Finally, in Section 2.4 we prove that $\operatorname{Ker}\left(\phi^{\Lambda^1(n,r)}_{\Lambda(n,r)}\right)$ is the strong idempotent ideal and that $C(\Lambda(n,r)) \cong S^+(n,r)$.

1 Skew product over monoids and strong idempotent ideals

1.1 Graded rings and modules

In this subsection we recollect results about graded algebras and graded modules, that were essentially proved in Eilenberg's paper [9].

Let Γ be a monoid with neutral element ϵ and A a Γ-graded associative algebra, that is

$$A = \bigoplus_{\gamma\in\Gamma} A_\gamma,$$

where A_γ is a subspace of A, for each $\gamma \in \Gamma$, and if $a_1 \in A_{\gamma_1}$ and $a_2 \in A_{\gamma_2}$ then $a_1 a_2 \in A_{\gamma_1\gamma_2}$. We will assume in addition that the unity e_A of A is an element of A_ϵ.

A left A-module M is Γ-graded if $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$, where each M_γ is a vector subspace of M, and for $a \in A_{\gamma_1}$ and $m \in M_{\gamma_2}$ we have $am \in M_{\gamma_1\gamma_2}$.

We will consider the category A-Γ-gr of left Γ-graded A-modules and A-module homomorphisms respecting grading, that is a map of A-modules $f\colon M_1 \to M_2$ is in A-Γ-gr if $f(M_{1,\gamma}) \subset M_{2,\gamma}$ for all $\gamma \in \Gamma$. Define the radical $\mathrm{rad}(A)$ of A as the intersection of all maximal graded left ideals of A. We also use the notion of ungraded radical $\mathrm{Rad}(A)$ of A, which is defined as the intersection of all (not-necessarily graded) maximal left ideals of A. In the following, we determine the conditions under which $\mathrm{rad}(A)$ and $\mathrm{Rad}(A)$ coincide.

Definition 1.1. We say that the monoid Γ is positive, if it has the property

$$\gamma_1\gamma_2 = \epsilon \Rightarrow \gamma_1 = \epsilon \text{ and } \gamma_2 = \epsilon.$$

Lemma 1.2. *Let Γ be a positive monoid and A a Γ-graded algebra. If m is a maximal graded left ideal of A, then*

$$m = m_\epsilon \oplus \bigoplus_{\gamma\neq\epsilon} A_\gamma$$

where m_ϵ is a maximal left ideal of A_ϵ.

Proof. Let n be a maximal left ideal of A_ϵ containing m_ϵ. It is clear, that

$$n' = n \oplus \bigoplus_{\gamma\neq\epsilon} A_\gamma$$

is a Γ-graded left ideal of A and that $m \subset n'$. Since m is maximal, it follows that $m = n'$. In particular, $n = m_\epsilon$. □

Corollary 1.3. *Suppose Γ is positive. Then $\bigoplus_{\gamma\neq\epsilon} A_\gamma$ is a subset of* $\mathrm{rad}(A)$.

Proof. By Lemma 1.2 $\bigoplus_{\gamma\neq\epsilon} A_\gamma$ is a subset of each maximal graded left ideal of A. □

Corollary 1.4. *Let Γ be a positive monoid and A a Γ-graded algebra. If m is a maximal graded left ideal of A, then m is a maximal left ideal of A in the ungraded sense.*

Proof. Since $\bigoplus_{\gamma\neq\epsilon} A_\gamma$ is an ideal of A, we have a surjective homomorphism $A \to A_\epsilon$. As m_ϵ is a maximal ideal of A_ϵ, the left A_ϵ-module A_ϵ/m_ϵ is simple. But then A_ϵ/m_ϵ is simple as an A-module. Since $A_\epsilon/m_\epsilon \cong A/m$, we get that m is a maximal left ideal of A. □

Corollary 1.5. *Suppose Γ is positive, then* $\mathrm{rad}(A) = \mathrm{Rad}(A_\epsilon) \oplus \bigoplus_{\gamma\neq\epsilon} A_\gamma$.

Proof. The radical $\mathrm{rad}(A)$ is a graded ideal of A, thus

$$\mathrm{rad}(A) = \bigoplus_{\gamma\in\Gamma} \mathrm{rad}(A)_\gamma.$$

Since by Corollary 1.3 $\bigoplus_{\gamma\neq\epsilon} A_\gamma$ is a subset of $\mathrm{rad}(A)$ we have that $\mathrm{rad}(A)_\gamma = A_\gamma$ for $\gamma \neq \epsilon$. Thus it is enough to check that $\mathrm{rad}(A)_\epsilon = \mathrm{Rad}(A_\epsilon)$.

Let m be a maximal left ideal of A_ϵ. Then

$$m' = m \oplus \bigoplus_{\gamma\neq\epsilon} A_\gamma$$

is a maximal graded left ideal of A. Now

$$\mathrm{rad}(A)_\epsilon = \mathrm{rad}(A) \cap A_\epsilon \subset m' \cap A_\epsilon = m.$$

As m was an arbitrary maximal left ideal of A_ϵ and $\mathrm{Rad}(A_\epsilon)$ is the intersection of all such left ideals, we get that $\mathrm{rad}(A)_\epsilon \subset \mathrm{Rad}(A_\epsilon)$.

Now, let m be a maximal graded left ideal of A. Then by Lemma 1.2

$$m = m_\epsilon \oplus \bigoplus_{\gamma\neq\epsilon} A_\gamma,$$

where m_ϵ is a maximal left ideal of A_ϵ. Therefore

$$\mathrm{Rad}(A_\epsilon) \subset m_\epsilon \subset m$$

and since m was an arbitrary maximal graded left ideal of A

$$\mathrm{Rad}(A_\epsilon) \subset \mathrm{rad}(A) \cap A_\epsilon = \mathrm{rad}(A)_\epsilon.$$

□

For $\gamma \in \Gamma$ we denote by $A(\gamma)$ the left Γ-graded module, defined by

$$A(\gamma)_{\gamma'} := A_{\gamma'\gamma}$$

and the action of A is given by multiplication:

$$\begin{aligned} A_{\gamma_1} \otimes A(\gamma)_{\gamma_2} &\to A(\gamma)_{\gamma_1\gamma_2} \\ a_1 \otimes a_2 &\mapsto a_1 a_2. \end{aligned}$$

Definition 1.6. We call a direct sum of modules of the form $A(\gamma)$ *a free* Γ-graded A-module. A direct summand in the category of Γ-graded A-modules of a free Γ-graded A-module is called *a projective* Γ*-graded* A*-module.*

Let P be a projective Γ-graded A-module generated by a single element $x \neq 0$ of degree $\gamma \in \Gamma$. Let F be a free Γ-graded module with a basis consisting of an element y of degree γ. Then there is an epimorphism $\phi\colon F \to P$, such that $\phi y = x$. Since P is projective, it follows that $\mathrm{Ker}(\phi)$ is a direct summand of F. Consequently there exists an idempotent $e \in A_\epsilon$ such that $P \cong Aex$. The degree γ of x is uniquely determined by P.

Definition 1.7. A left Γ-graded A-module is called *quasi-free* if it is a direct sum of projective modules each of which is generated by a single (homogeneous) element.

In the following we assume, that

- the monoid Γ is positive;
- the ring $\bar{A} = A/\operatorname{rad}(A) = A_\epsilon/\operatorname{Rad}(A_\epsilon)$ is semi-simple;
- each idempotent in $\bar{A}$ can be lifted to A_ϵ.

Following Eilenberg [9] we say that the subcategory $\mathscr{C}$ of A-Γ-gr is *perfect* if

(i) $\mathscr{C}$ is full.

(ii) If $\phi\colon M \to N$ is an epimorphism and $M \in \mathscr{C}$ then $N \in \mathscr{C}$.

(iii) $A \in \mathscr{C}$.

(iv) If P is quasi-free and $P/\operatorname{rad}(A)P \in \mathscr{C}$, then $P \in \mathscr{C}$.

(v) If $M \in \mathscr{C}$ and $\operatorname{rad}(A)M = M$, then $M = 0$.

Suppose $\mathscr{C}$ is a perfect subcategory of A-Γ-gr.

Definition 1.8. An epimorphism $\phi\colon P \to M$ in $\mathscr{C}$ is called *minimal* if P is projective and $\operatorname{Ker}(\phi) \subset \operatorname{rad}(A)P$.

Proposition 1.9. *Every $M \in \mathscr{C}$ admits a minimal epimorphism $\phi\colon P \to M$. If $\phi'\colon P' \to M$ is another minimal epimorphism, then there exists a homomorphism $\pi\colon P \to P'$ such that $\phi'\pi = \phi$, and each such homomorphism is an isomorphism.*

Proof. The proof is word by word repetition of the proof of [9, Proposition 3], with understanding that all $\mathbb{N}$-graded modules have to be replaced by Γ-graded modules. $\square$

Suppose further, that $\mathscr{C}$ satisfies the additional condition

- If $M \in \mathscr{C}$ and $N \subset M$, then $N \in \mathscr{C}$.

Then, as usual, by iterating the minimal epimorphism construction we get for each $M \in \mathscr{C}$ a projective resolution

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

such that $\operatorname{Im}(d_i) \subset \operatorname{rad}(A)P_i$ for $i = 0, 1, \ldots$. This projective resolution is called *minimal.*

The following two results are formal consequences of the definition and Proposition 1.9. For proofs the reader is referred to [9, Proposition 7 and Theorem 8].

Proposition 1.10. *Let $P_\bullet$ and $P'_\bullet$ be minimal projective resolutions of a module $M \in \mathscr{C}$. Then there exists a map $f \colon P_\bullet \to P'_\bullet$ over the identity map of M and each such map is an isomorphism.*

Theorem 1.11. *Let $M \in \mathscr{C}$ and let $P_\bullet$ be a projective resolution of M. Then $P_\bullet$ decomposes into a direct sum $P_\bullet = \bar{P}_\bullet \oplus W_\bullet$ of subcomplexes such that $\bar{P}_\bullet$ is a minimal projective resolution of M, while W is a projective resolution of the zero module.*

From now on we restrict ourselves to the case when Γ is an ordered monoid and the neutral element ϵ of Γ is the least element of Γ. Note that all such monoids are positive, since

$$\gamma_1 \neq \epsilon,\ \gamma_2 \neq \epsilon \Rightarrow \gamma_1 > \epsilon,\ \gamma_2 > \epsilon \Rightarrow \gamma_1\gamma_2 > \epsilon \Rightarrow \gamma_1\gamma_2 \neq \epsilon.$$

A Γ-graded A-module M is said to be *locally finitely generated* if M is generated by a set X of (homogeneous) elements such that the sets $X \cap M_\gamma$ are finite, for all $\gamma \in \Gamma$.

Theorem 1.12. *The category $\mathscr{C}(\Gamma, A)$ of all locally finitely generated Γ-graded left A-modules is a perfect subcategory of A-Γ-gr. It is closed under taking subobjects if and only if A_ϵ is a left Noetherian ring and each A_γ is finitely generated as a left A_ϵ-module.*

Proposition 1.13. *Let A be a finite dimensional Γ-graded algebra. Then the ungraded radical* $\mathrm{Rad}(A)$ *of A coincides with the graded radical* $\mathrm{rad}(A)$ *of A.*

Proof. From Corollary 1.4 it follows that $\mathrm{Rad}(A)$ is a subset of $\mathrm{rad}(A)$.

We shall show, that $\mathrm{rad}(A)$ is nilpotent. Since $\mathrm{Rad}(A)$ is a maximal nilpotent ideal of A, we shall get that $\mathrm{rad}(A) = \mathrm{Rad}(A)$.

By Corollary 1.5, we have $\mathrm{rad}(A) = \mathrm{Rad}(A_e) \oplus \bigoplus_{\gamma \neq e} A_\gamma$. Denote by N the ideal $\bigoplus_{\gamma \neq e} A_\gamma$. Then N is nilpotent. In fact, we show, that if $\dim(A) = n$, then the product of any $n+1$ elements of N is zero. Clearly, this should be checked only for homogeneous elements. Let $a_0, a_1, \dots a_n$ be a sequence of homogeneous elements from N. Suppose, that for each i the element a_i is from A_{γ_i}. Then, since each $\gamma_i > \epsilon$ we have a strictly increasing sequence

$$\gamma_0 < \gamma_0\gamma_1 < \cdots < \gamma_0\gamma_1 \dots \gamma_n.$$

As A is n-dimensional one of the $n+1$ spaces

$$A_{\gamma_0}, A_{\gamma_0\gamma_1}, \dots, A_{\gamma_0\gamma_1\dots\gamma_n}$$

should be zero. In particular, one of the products

$$a_{\gamma_0}, a_{\gamma_0}a_{\gamma_1}, \dots, a_{\gamma_0}a_{\gamma_1}\dots a_{\gamma_n}$$

is zero. Thus $a_{\gamma_0}a_{\gamma_1}\dots a_{\gamma_n} = 0$ and $N^{n+1} = 0$.

Denote $\mathrm{Rad}(A_\epsilon)$ by M. For any natural numbers $k_0, \ldots, k_{n+1}$, we have

$$M^{k_0} N M^{k_1} \ldots M^{k_n} N M^{k_{n+1}} = \left(M^{k_0} N\right)\left(M^{k_1} N\right)\ldots\left(M^{k_{n-1}} N\right)\left(M^{k_n} N M^{k_{n+1}}\right) \subset N^{n+1} = 0.$$

As $\mathrm{Rad}(A_\epsilon)$ is a nilpotent ideal of the algebra A_ϵ, there is m, such that $(\mathrm{Rad}(A_\epsilon))^m = M^m = 0$. Then

$$(M+N)^{nm+n} = \sum_{l=0}^{n-1} \sum_{(k_0,\ldots,k_l)\colon \sum k_i = nm+n-l} M^{k_0} N M^{k_1} \ldots M^{k_{l-1}} N M^{k_l} = 0,$$

since in each summand at least one k_i is greater than m. $\square$

1.2 Skew product over monoids

We say that an algebra B is a Γ-algebra, if there is a given right action

$$\begin{aligned} r\colon B \times \Gamma &\to B \\ (b, \gamma) &\mapsto b^\gamma \end{aligned}$$

such that for each $\gamma \in \Gamma$ the map

$$\begin{aligned} B &\to B \\ b &\mapsto b^\gamma \end{aligned}$$

is an algebra homomorphism.

Let A be a Γ-graded algebra. We define the interchange map $T\colon B \otimes A \to A \otimes B$ by

$$\begin{aligned} B \otimes A_\gamma &\to A_\gamma \otimes B \\ b \otimes a_\gamma &\mapsto a_\gamma \otimes b^\gamma \end{aligned}$$

and a binary operation m on $A \otimes B$ by

$$m\colon A \otimes B \otimes A \otimes B \xrightarrow{1_A \otimes T \otimes 1_B} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

Denote the vector space $A \otimes B$ with the binary operation m on it by $A \ltimes_\Gamma B$.

Proposition 1.14. *$A \ltimes_\Gamma B$ is an algebra.*

Proof. We have to check that $e_A \otimes e_B$ is a neutral element with respect to m and that m is an associative operation. This follows from the following computation

$$e_A \otimes e_B \otimes a_\gamma \otimes b \mapsto e_A \otimes a_\gamma \otimes e_B^\gamma \otimes b \mapsto a_\gamma \otimes b,$$

$$a_\gamma \otimes b \otimes e_A \otimes e_B \mapsto a_\gamma \otimes e_A \otimes b^e \otimes e_B \mapsto a_\gamma \otimes b$$

and

$$\begin{tikzcd}
a_{\gamma_1} \otimes b_1 \otimes a_{\gamma_2} \otimes b_2 \otimes a_{\gamma_3} \otimes b_3 \arrow[r, mapsto] \arrow[dd, mapsto] & a_{\gamma_1} a_{\gamma_2} \otimes b_1^{\gamma_2} b_2 \otimes a_{\gamma_3} \otimes b_3 \arrow[d, mapsto] \\
 & a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} \otimes (b_1^{\gamma_2} b_2)^{\gamma_3} b_3 \arrow[d, equal] \\
a_{\gamma_1} \otimes b_1 \otimes a_{\gamma_2} a_{\gamma_3} \otimes b_2^{\gamma_3} b_3 \arrow[r, mapsto] & a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} \otimes b_1^{\gamma_2 \gamma_3} b_2^{\gamma_3} b_3.
\end{tikzcd}$$

$\square$

Note, that the embedding $A \to A \ltimes_\Gamma B$ given by

$$a \mapsto a \otimes 1_B$$

is a homomorphism of algebras. In the following we will consider the elements of A as elements of $A \ltimes_\Gamma B$ through this embedding.

The algebra $A \ltimes_\Gamma B$ is itself Γ-graded. In fact

$$A \ltimes_\Gamma B = \bigoplus_{\gamma \in \Gamma} (A_\gamma \otimes B).$$

Let N be a B-module and $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ a Γ-graded A-module. We define a $(A \ltimes_\Gamma B)$-module structure on $M \otimes N$ as follows

$$a_{\gamma_1} \otimes b \otimes m_{\gamma_2} \otimes n \mapsto a_{\gamma_1} m_{\gamma_2} \otimes b^{\gamma_2} n$$

for all $a_{\gamma_1} \in A_{\gamma_1}$, $b \in B$, $m_{\gamma_2} \in M_{\gamma_2}$ and $n \in N$. We denote this module by $M \ltimes_\Gamma N$. This is a Γ-graded module, with $(M \ltimes_\Gamma N)_\gamma = M_\gamma \otimes N$.

Let $\varphi\colon M_1 \to M_2$ be a homomorphism of Γ-graded A-modules and $\psi\colon N_1 \to N_2$ a homomorphism of B-modules. We denote by $\varphi \ltimes_\Gamma \psi$ the map

$$\begin{aligned} M_1 \ltimes_\Gamma N_1 &\to M_2 \ltimes_\Gamma N_2 \\ m \otimes n &\mapsto \varphi(m) \otimes \psi(n). \end{aligned}$$

Proposition 1.15. *The map $\varphi \ltimes_\Gamma \psi$ is a homomorphism of Γ-graded $A \ltimes_\Gamma B$-modules.*

Proof. Since for all $m_\gamma \in (M_1)_\gamma$, the element $\varphi(m_\gamma)$ is an element of $(M_2)_\gamma$, it follows that $\varphi \ltimes_\Gamma \psi(m_\gamma \otimes n) = \varphi(m_\gamma) \otimes \psi(n)$ is an element of $(M_2)_\gamma \otimes N$. Thus $\varphi \ltimes_\Gamma \psi$ preserves the Γ-grading.

That $\varphi \ltimes_\Gamma \psi$ is a $A \ltimes_\Gamma B$-homomorphism follows from the following compu-

tation

$$\begin{tikzcd}
a_{\gamma_1} \otimes b \otimes m_{\gamma_2} \otimes n \arrow[r, mapsto] \arrow[dd, mapsto] & a_{\gamma_1} m_{\gamma_2} \otimes b^{\gamma_2} n \arrow[d] \\
& \phi(a_{\gamma_1} m_{\gamma_2}) \otimes \psi(b^{\gamma_2} n) \arrow[d, equal] \\
a_{\gamma_1} \otimes b \otimes \varphi(m_{\gamma_2}) \otimes \psi(n) \arrow[r, mapsto] & a_{\gamma_1} \varphi(m_{\gamma_2}) \otimes b^{\gamma_2} \psi(n),
\end{tikzcd}$$

where $a_{\gamma_1} \in A_{\gamma_1}$ and $m_{\gamma_2} \in M_{\gamma_2}$. □

It follows from these results that the correspondence

$$(M, N) \mapsto M \ltimes_\Gamma N$$
$$(\varphi, \psi) \mapsto \varphi \ltimes_\Gamma \psi$$

gives a bifunctor from the categories A-Γ-gr and B-mod to the category $(A \ltimes_\Gamma B)$-Γ-gr. In particular, for each B-module N we have the functor $- \ltimes_\Gamma N$ from the category A-Γ-gr to the category $(A \ltimes_\Gamma B)$-Γ-gr. This functor is obviously exact, since it is just a tensor product with N on the level of underlying vector spaces.

Proposition 1.16. *Suppose, that Γ acts by automorphisms on B. Let F_A be a Γ-graded free A-module, and F_B a free B-module. Then the module $F_A \ltimes_\Gamma F_B$ is a Γ-graded free $A \ltimes_\Gamma B$-module.*

Proof. Let $\{v_\alpha \mid \alpha \in I\}$ be a Γ-homogeneous A-basis of F_A and $\{w_\beta \mid \beta \in J\}$ a B-basis of F_B. We shall show, that $\{v_\alpha \otimes w_\beta \mid \alpha \in I,\ \beta \in J\}$ is a Γ-homogeneous $A \ltimes_\Gamma B$-basis of $F_A \ltimes_\Gamma F_B$.

First we show that each element in $F_A \ltimes_\Gamma F_B$ can be written as a $A \ltimes_\Gamma B$-combination of elements from $\{v_\alpha \otimes w_\beta \mid \alpha \in I,\ \beta \in J\}$. Clearly, it should be checked only for elements of the form $u \otimes v$ with $u \in A$ and $v \in B$. Since $\{v_\alpha \mid \alpha \in I\}$ is a basis of A we can write u as a linear combination

$$u = \sum_{\alpha \in I} x_\alpha v_\alpha,$$

with $x_\alpha \in A$. Since $\{w_\beta \mid \beta \in J\}$ is a basis of B we can write v as a linear combination

$$v = \sum_{\beta \in J} y_\beta w_\beta,$$

with $y_\beta \in B$. Denote by γ_α degree of v_α. Recall, that Γ acts by automorphisms on B and therefore the map $\gamma^{-1}\colon B \to B$ is well defined for all $\gamma \in \Gamma$. Then

$$u \otimes v = \sum_{\alpha \in I} \sum_{\beta \in B} x_\alpha v_\alpha \otimes y_\beta w_\beta = \sum_{\alpha \in I} \sum_{\beta \in J} (x_\alpha \otimes \gamma_\alpha^{-1}(y_\beta))(v_\alpha \otimes w_\beta).$$

Now, we show that the set $\{v_\alpha \otimes w_\beta \mid \alpha \in I,\ \beta \in J\}$ is linearly independent over $A \ltimes_\Gamma B$. Let $\{a_\theta\}$ be a homogeneous $\mathbb{K}$-basis of A, and $\{b_\mu\}$ a $\mathbb{K}$-basis of B. Since Γ acts by automorphisms on B, for each $\gamma \in \Gamma$ there are elements $b_{\mu,\gamma}$ such that $\gamma(b_{\mu,\gamma}) = b_\mu$ and the set $\{b_{\mu,\gamma}\}$ is a $\mathbb{K}$-basis of B. Thererfore, for each $\gamma \in \Gamma$ the set $\{a_\theta \otimes b_{\mu,\gamma}\}$ is a $\mathbb{K}$-basis of $A \ltimes_\Gamma B$. Suppose, that

$$\sum_{\alpha \in I} \sum_{\beta \in J} \kappa_{\alpha,\beta} v_\alpha \otimes w_\beta = 0,$$

where $\kappa_{\alpha,\beta} \in A \ltimes_\Gamma B$. Then there are elements $\eta_{\alpha,\beta,\theta,\mu}$ of $\mathbb{K}$ such that

$$\kappa_{\alpha,\beta} = \sum_\theta \sum_\mu \eta_{\alpha,\beta,\theta,\mu} a_\theta \otimes b_{\mu,\gamma_\alpha}.$$

Thus

$$0 = \sum_{\alpha \in I} \sum_{\beta \in J} \kappa_{\alpha,\beta} v_\alpha \otimes w_\beta = \sum_{\alpha \in I, \beta \in J} \sum_{\theta,\mu} \eta_{\alpha,\beta,\theta,\mu} (a_\theta \otimes b_{\mu,\gamma_\alpha})(v_\alpha \otimes w_\beta)$$
$$= \sum_{\alpha \in I, \beta \in J} \sum_{\theta,\mu} \eta_{\alpha,\beta,\theta,\mu} a_\theta v_\alpha \otimes \gamma_\alpha(b_{\mu,\gamma_\alpha}) w_\beta = \sum_{\alpha \in I, \beta \in J} \sum_{\theta,\mu} \eta_{\alpha,\beta,\theta,\mu} a_\theta v_\alpha \otimes b_\mu w_\beta.$$

Since $\{a_\theta v_\alpha\}$ is a $\mathbb{K}$-basis of F_A and $\{b_\mu w_\beta\}$ is a $\mathbb{K}$-basis of F_B, it follows, that $\{a_\theta v_\alpha \otimes b_\mu w_\beta\}$ is a $\mathbb{K}$-basis of $F_A \ltimes_\Gamma F_B$. Therefore, all $\eta_{\alpha,\beta,\theta,\mu}$ are zero and consequently all $\kappa_{\alpha,\beta}$ are zero.

□

Proposition 1.17. *Suppose, that Γ acts by automorphisms on B. Let P be a Γ-graded projective A-module, and N a projective B-module. Then the module $M \ltimes_\Gamma N$ is a Γ-graded projective $A \ltimes_\Gamma B$-module.*

Proof. Since P is a Γ-graded projective A-module, it is a direct summand of a free module F_A over A. We denote the corresponding inclusion and projection Γ-graded A-module homomorphisms by i_P and π_P respectively. Analogously, since N is a projective B-module, it is a direct summand of some free B-module F_B. We denote the respective inclusion and projective homomorphisms by i_N and π_N. Then we have maps

$$\begin{aligned} i\colon M \ltimes_\Gamma N &\to F_A \ltimes_\Gamma F_B \\ m \otimes n &\mapsto i_M(m) \otimes i_N(m) \end{aligned}$$

and

$$\begin{aligned} \pi\colon F_A \ltimes_\Gamma F_B &\to M \ltimes_\Gamma N \\ f \otimes g &\mapsto \pi_M(f) \otimes \pi_N(g). \end{aligned}$$

The maps i and π are Γ-graded $A \ltimes_\Gamma B$-module homomorphism by Proposition 1.15. It is obvious that $\pi \circ i = 1_{M \ltimes_\Gamma N}$. Thus, $M \ltimes_\Gamma N$ is a direct summand of $F_A \ltimes_\Gamma F_B$. But by Theorem 1.16 the module $F_A \ltimes_\Gamma F_B$ is a free Γ-graded $A \ltimes_\Gamma B$-module.

□

Let N be a projective B-module. Then Proposition 1.17 shows, that the functor $- \ltimes_\Gamma N$ preserves projective resolutions. Note, that it does not map in general a minimal projective resolution into a minimal projective resolution.

Proposition 1.18. *Suppose that* Γ *acts by automorphisms on* B*. If* $A_\epsilon \cong \mathbb{K}$ *and* $\mathrm{Rad}(B) = 0$*, then*

$$\mathrm{rad}(A \ltimes_\Gamma B) = \mathrm{rad}(A) \ltimes_\Gamma B.$$

Proof. Note, that $A_\epsilon \ltimes_\Gamma B$ is a subalgebra of $A \ltimes_\Gamma B$ and it is isomorphic to the usual tensor product of algebras $A_\epsilon \otimes B$. By Corollary 1.5, we have

$$\begin{aligned}
\mathrm{rad}(A \ltimes_\Gamma B) &= \mathrm{Rad}((A \ltimes_\Gamma B)_\epsilon) \oplus \bigoplus_{\gamma \neq \epsilon} (A \ltimes_\Gamma B)_\gamma \\
&= \mathrm{Rad}(A_\epsilon \otimes B) \oplus \left(\bigoplus_{\gamma \neq \epsilon} A_\gamma \right) \ltimes_\Gamma B \\
&= \mathrm{Rad}(\mathbb{K} \otimes B) \oplus \mathrm{rad}(A) \ltimes_\Gamma B = \mathrm{rad}(A) \ltimes_\Gamma B.
\end{aligned}$$

□

Corollary 1.19. *Suppose* $A_\epsilon \cong \mathbb{K}$ *and* $\mathrm{Rad}(B) = 0$*. Let* $\phi\colon M_1 \to M_2$ *be a homomorphism of* Γ*-graded* A*-modules. Suppose, that* $\mathrm{Im}(\phi) \subset \mathrm{rad}(A)M_2$*. Then for any* B*-module* N*, we have*

$$\mathrm{Im}(\phi \ltimes_\Gamma N) \subset \mathrm{rad}(A \ltimes_\Gamma B)(M_2 \ltimes_\Gamma N).$$

Proof. We have

$$\mathrm{Im}(\phi \ltimes_\Gamma N) = \mathrm{Im}(\phi) \ltimes_\Gamma N \subset \mathrm{rad}(A)M_2 \ltimes_\Gamma N.$$

From Proposition 1.18 we have

$$\mathrm{rad}(A \ltimes_\Gamma B)(M_2 \ltimes_\Gamma N) = (\mathrm{rad}(A) \ltimes B)(M_2 \ltimes_\Gamma N).$$

As Γ acts by automorphism on B, for every $\gamma \in \Gamma$ we have $B^\gamma = B$. Therefore

$$(\mathrm{rad}(A) \ltimes B)(M_2 \ltimes_\Gamma N) = \mathrm{rad}(A)M_2 \ltimes_\Gamma BN = \mathrm{rad}(A)M_2 \ltimes_\Gamma N.$$

□

1.3 Strong idempotent ideals

Let A be an algebra and I a two-sided ideal of A. In this section we give an overview of results from [4] concerning the inclusion functor $A/I - mod \to A - mod$.

We always have a map $\phi^i_{X,Y}\colon \mathrm{Ext}^I_{A/I}(X,Y) \to \mathrm{Ext}^I_A(X,Y)$ for $i \geq 0$ and X, Y in $A/I-mod$, induced by the canonical isomorphism $\phi^0_{x,y}\colon \mathrm{Hom}_{A/I}(X,Y) \to$

$\mathrm{Hom}_A(X, Y)$. Analogously, there are maps $\psi^I_{X,Y}\colon \mathrm{Tor}^A_i(Y, X) \to \mathrm{Tor}^{A/I}_i(Y, X)$ for $i \geq 0$ and X left and Y right A/I-module, induced by the canonical isomorphism $Y \otimes_A X \cong Y \otimes_{A/I} X$. In [4, Proposition 1.2 and Proposition 1.3], there was proved the equivalence of the following properties for I a two-sided ideal of an algebra A and k a natural number:

(i) $\phi^i_{X,Y}\colon \mathrm{Ext}^i_{A/I}(X, Y) \to \mathrm{Ext}^i_A(X, Y)$ is an isomorphism for all X, Y in $A/I - mod$ and all $1 \leq i \leq k$.

(ii) $\mathrm{Ext}^i_A(A/I, Y) = 0$ for all A/I-module Y and all $1 \leq i \leq k$.

(iii) $\psi^i_{X,Y}\colon \mathrm{Tor}^{A/I}_i(X, Y) \to \mathrm{Tor}^A_i(X, Y)$ is an isomorphism for all X in $(A/I)^{op}- mod$ and Y in $A/I - mod$ and all $0 \leq i \leq k$.

(iv) $\mathrm{Tor}^A_i(A/I, Y) = 0$ for all Y in A/I-mod and all $1 \leq i \leq k$.

(v) if Y is an A/I-module and $\cdots \to P_1 \to P_0 \to Y \to 0$ is a minimal projective resolution of Y in $A - mod$, then

$$P_k/IP_k \to \cdots \to P_0/IP_0 \to Y \to 0$$

is the beginning of a minimal projective resolution of Y in A/I-mod.

Definition 1.20. If one of the above conditions holds, we say that I is a k-*idempotent ideal*. If the conditions hold for all $k \in \mathbb{N}$, then we say that I is *strong idempotent ideal*.

We have the following obvious property of k-idempotent ideals

Proposition 1.21. *If*

$$I_0 \subset I_1 \subset \cdots \subset I_l \subset A$$

is a chain of ideals in A, such that for all j the ideal I_j/I_{j-1} of A/I_{j-1} is k-idempotent, then I_l is k-idempotent ideal of A

Note that from [4, lemma 1.4(a) and Proposition 4.6] follows that an ideal I is 1-idempotent if and only if $I = AeA$ for some idempotent $e \in A$.

Proposition 1.22. *Let $e \in A$ be an idempotent. An ideal AeA is 2-idempotent if and only if the map induced by the multiplication in A*

$$Ae \otimes_{eAe} eA \to AeA$$

is an isomorphism.

Proof. It follows from [4, lemma 1.4(b), and Proposition 4.6]. □

Definition 1.23. Let $e \in A$ be an idempotent. An ideal $I = AeA$ is called an *heredity* ideal if

(i) $e\,\mathrm{rad}(A)e = 0$;

(ii) I considered as a left A-module is projective.

Proposition 1.24. *Suppose that $I = AeA$ and I_A is projective as an A-module. Then I is a strong idempotent ideal.*

Proof. It follows from the definition of strong idempotent ideal and [6, Statement 3]. ☐

Corollary 1.25. *If I is a heredity ideal, then I is strong idempotent.*

Proposition 1.26. *An ideal $I = AeA$ is a heredity ideal if and only if*

(i) I is 2-idempotent;

(ii) $e\,\mathrm{rad}(A)e = 0$.

Proof. It follows from [6, Statement 7]. ☐

Corollary 1.27. *If $I = AeA$ is 2-idempotent, and $e\,\mathrm{rad}(A)e = 0$, then I is strong idempotent.*

1.4 Criterion of heredity

In this section Γ is always an ordered positive monoid and A a Γ-graded algebra. Let S be a Γ-set, that is a set where Γ acts by endomorphisms. Set $B = \mathrm{Maps}(S, \mathbb{K})$. Then B is a Γ-algebra and we can consider the skew product algebra

$$C = A \ltimes_\Gamma B.$$

For simplicity, if $a \in A$ and $b \in B$ we will sometimes write ab for the element $a \otimes b$ of C. For each subset X of S there is an idempotent in C

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases}$$

Define the algebra $C(X)$ by

$$C(X) = C/C\chi_{\bar{X}}C,$$

where $\bar{X}$ denotes the complement of X in S. In this section, we will prove some general results concerning the algebras $C(X)$.

We say that Γ acts on effectively, if $\gamma_1 x \neq \gamma_2 x$ for all $x \in S$ and all $\gamma_1 \neq \gamma_2$ from Γ. From now on we will assume, that Γ acts effectively on S. We introduce a partial order on S, by

$$x \leq_\Gamma y \Leftrightarrow \exists \gamma \in \Gamma : y = \gamma x.$$

Definition 1.28. We say that $X \subset S$ is convex if for all $x, y \in X$ it contains all $z \in S$ which lie between x and y.

Proposition 1.29. *Let X be a convex subset of S. Then*

$$C(X) \cong \chi_X C \chi_X.$$

Proof. Note, that we always have a surjective homomorphism

$$\begin{aligned} \pi : &\chi_X C \chi_X \to C(X) \\ &a \otimes b \mapsto [a \otimes b]. \end{aligned}$$

Let $x, y \in X$ and suppose that $y = \gamma x$ for some $\gamma \in \Gamma$. Then

$$\chi_y C \chi_{\bar{X}} C \chi_x = \left\langle \chi_y a_1 \chi_z a_2 \chi_x | \gamma = \gamma_2 \gamma_1, z = \gamma_1 x, z \in \bar{X}, a_1 \in A_{\gamma_1}, a_2 \in A_{\gamma_2} \right\rangle.$$

But if z lies between x and y, then $z \in X$ and the above set is empty. Therefore the restriction of π on $\chi_y C \chi_x$ is injective, and since

$$\chi_X C \chi_X = \bigoplus_{x,y \in X} \chi_y C \chi_x,$$

the map π is injective. □

Corollary 1.30. *Let J be a homogeneous basis of A and Y a Γ-convex subset of S. Then*

$$\{a \chi_y | a \in J;\ y, \deg(a) y \in Y\},$$

is a basis of $C(Y)$.

Let $X \subset Y$ be two subsets of S. Then we have a surjective homomorphism

$$C(Y) \to C(X).$$

We give a criterion for projectivity of $C(X)$ over $C(Y)$.

Suppose $A = A_1 \otimes A_2$ is a product of two Γ-graded algebras, that is

$$A_\gamma = \bigoplus_{\gamma = \gamma_1 \gamma_2} (A_1)_{\gamma_1} \otimes (A_2)_{\gamma_2}.$$

For $i = 1, 2$, let Γ_i be submonoid of Γ such that for $\gamma \notin \Gamma_i$ the space $(A_i)_\gamma$ is zero. Denote by C_i the skew product $A_i \ltimes_{\Gamma_i} B$.

Theorem 1.31. *Let J_1 and J_2 be homogeneous bases of A_1 and A_2, respectively. Suppose that there are subsets I_y of J_2 for each $y \in Y$, such that the set*

$$\{a \otimes \chi_y | a \in I_y,\ y \in Y\}$$

is a basis of $C_2(Y)$ and the set

$$\{a_1 a_2 \otimes \chi_x | a_1 \in J_1,\ a_2 \in I_y,\ y \in Y,\ \deg(a_1 a_2) y \in Y\}$$

is a basis of $C(Y)$. Denote by Z the difference $Y \setminus X$ and by e the idempotent χ_Z of $C(Y)$. If

(i) $\Gamma_1 X \cap Y = X$;

(ii) $\Gamma_2 Z \cap Y = Z$,

then

(i)
$$C(Y)e \otimes_{eC(Y)e} eC(Y) \cong C(Y)eC(Y);$$

(ii) the set

$$\{a_1a_2 \otimes \chi_x | a_1 \in J_1,\ a_2 \in I_x; x, \deg(a_2)x, \deg(a_1a_2)x \in X\}$$

is a basis of $C(X)$.

Proof. For $i = 1, 2$ denote by $C_i(Y)$ the algebras

$$C_i / C_i \chi_{\bar{Y}} C_i.$$

The proof of the theorem is a consequence of the following two lemmata.

Lemma 1.32. *The set*

$$J = \{a_1a_2 \otimes \chi_y | a_1 \in J_1,\ a_2 \in I_y,\ y \in Y,\ \deg(a_1a_2)y \in Y,\ \deg(a_2)y \in Z\},$$

is a basis of $C(Y)eC(Y)$.

Proof. Note, that every element of J is an element of $C(Y)eC(Y)$. In fact, denote $\deg(a_2)$ by γ, then

$$a_1a_2 \otimes \chi_y = a_1 \otimes \chi_{\gamma y} \cdot a_2 \in C(Y)\chi_{\gamma y}C(Y) \subset C(Y)eC(Y).$$

Moreover, J is a subset of a basis of $C(Y)$. Thus, it is enough to check that J generates $C(Y)eC(Y)$. It is clear that $C(Y)eC(Y)$ is generated by the set

$$\tilde{J} = \left\{ a_1'a_2' \otimes \chi_z \cdot a_1a_2 \otimes \chi_y \;\middle|\; \begin{array}{l} a_1, a_1' \in J_3; \\ a_2 \in I_y, a_2' \in I_z;\ y \in Y \\ \deg(a_1'a_2')z \in Y \\ \deg(a_1a_2)y = z \in Z \end{array} \right\}.$$

Let $a_1'a_2' \otimes \chi_z \cdot a_1a_2 \otimes \chi_y$ be an element of $\tilde{J}$. Denote by γ_i the degree of a_i. Then $\gamma_1\gamma_2 y$ is an element of Z. From the conditions of theorem it follows, that $z = \gamma_2 y$ is an element of Z. In fact, assume $z \in X$, then $\gamma_1 z \in X$ and thus $\gamma_1\gamma_2 y \in X$. Contradiction.

Now, the product

$$a_1'a_2' \otimes \chi_z a_1 = a_1'a_2'a_1 \otimes \chi_z$$

can be written as a linear combination of elements

$$a_1''a_2'' \otimes \chi_z$$

with $a_1'' \in J_1$, $a_2'' \in I_z$, since the set

$$\{a_1 a_2 \otimes \chi_x | a_1 \in J_1,\ a_2 \in I_y,\ y \in Y\}$$

is a basis of $C(Y)$. And the products

$$a_2'' \otimes \chi_z \cdot a_2 \otimes \chi_x = a_2'' a_2 \otimes \chi_x$$

can be written as linear combinations of $a_2''' \otimes \chi_x$, where $a_2''' \in I_x$, since the set

$$\{a \otimes \chi_y | a \in I_y,\ y \in Y\}$$

is a basis of $C_1(Y)$. Moreover, in each case,

$$\deg(a_2''')y = \deg(a_2'')\deg(a_2)y = \deg(a_2'')z' \in \Gamma_2 Z \cap Y = Z$$

and

$$\deg(a_1'')\deg(a_2''')y = \deg(a_1'')\deg(a_2'')z' = \deg(a_1'a_2')\deg(a_1)z' = \deg(a_1'a_2')z \in Y.$$

Thus every element of $\tilde{J}$ can be written as a linear combination of elements from J. $\square$

Lemma 1.33. *The set*

$$I = \{(a_1 \otimes \chi_z) \otimes (a_2 \otimes \chi_y) | a_1 \in J_1,\ a_2 \in I_y,\ \deg(a_2)y = z \in Z,\ \deg(a_1 a_2)y \in Y\}$$

is a basis of $C(Y)e \otimes_{eC(Y)e} eC(Y)$.

Proof. It is clear that all elements of I are elements of $C(Y)e \otimes_{eC(Y)e} eC(Y)$. Since $\pi(I) = J$ is a basis of $C(Y)\chi_Z C(Y)$, it follows that the elements of I are linear independent. We show, that every element of $C(Y)e \otimes_{eC(Y)e} eC(Y)$ can be written as a linear combination of elements from I.

It is clear that $C(Y)e \otimes_{eC(Y)e} eC(Y)$ is generated by the set

$$\tilde{I} = \left\{ (a_1'a_2' \otimes \chi_z) \otimes (a_1 a_2) \otimes \chi_y \,\middle|\, \begin{array}{c} a_1', a_1 \in J_1; a_2' \in I_z; a_2 \in I_y; \\ y \in Y; z = \deg(a_1 a_2)y \in Z \end{array} \right\}.$$

Now $a_1 a_2 \otimes \chi_y = a_1 \otimes \chi_{y'} \cdot a_2 \otimes y$ and $y' \in Z$, because otherwise

$$z = \deg(a_1)\deg(a_2)y = \deg(a_1)y' \in \Gamma_1 X \cap Y = X,$$

that contradicts to the condition $z \in Z$. Therefore $a_1 \otimes \chi_{y'} = \chi_z \cdot a_1 \cdot \chi_{y'}$ is an element of $eC(Y)e$. Thus

$$(a_1'a_2' \otimes \chi_z) \otimes (a_1 a_2 \otimes \chi_y) = (a_1'a_2'a_1 \otimes \chi_{y'}) \otimes (a_2 \otimes \chi_y).$$

Now $a_1'a_2'a_1 \otimes \chi_{y'}$ can be written as a linear combination of elements of the form $a_1''a_2'' \otimes \chi_{y'}$, with $a_1'' \in J_1$ and $a_2'' \in I_{y'}$. Since $y' \in Z$, we have that $\deg(a_2'')y' \in \Gamma_2 Z \cap Y = Z$. Therefore, $a_2''\chi_{y'} \in eC(Y)e$ and

$$(a_1''a_2'' \otimes \chi_{y'}) \otimes (a_2 \otimes \chi_y) = (a_1'' \otimes \chi_{y'}) \otimes (a_2''a_2 \otimes \chi_y).$$

Now, since $\{a_2 \otimes \chi_y | a_2 \in I_y,\ y \in Y\}$ is a basis of $C_2(Y)$, we can write $a_2''a_2 \otimes \chi_y$ as a linear combination of elements of the form $a_2''' \otimes \chi_y$, where $a_2''' \in I_y$, moreover $\deg(a_2''')y = \deg(a_2'')\deg(a_2)y = \deg(a_2'')\chi_{y'} \in \Gamma_2 Z \cap Y = Z$. $\square$

From two previous lemmata it follows that π gives a bijection between the basis I of $C(Y)e \otimes_{eC(Y)e} eC(Y)$ and the basis J of $C(Y)eC(Y)$. Therefore π is an isomorphism.

The last claim of the theorem follows from the description of bases of $C(Y)$ and $C(Y)eC(Y)$.

□

Corollary 1.34. *Under the same conditions as in Theorem 1.31, suppose* $Z = \{z_1, z_2, \dots, z_m\}$ *and*

(i) $i < j$, *if* $z_i \leq_{\Gamma_2} z_j$;

(ii) $j < i$, *if* $z_i \leq_{\Gamma_1} z_j$.

Denote by X_k *the set* $X \cup \{z_1, \dots, z_k\}$ *and by* e_k *the idempotent* χ_{z_k} *in* $C(X_k)$. *Then the ideal* $C(X_k)e_kC(X_k)$ *is 2-idempotent.*

If additionally $\mathrm{Rad}(A_\epsilon) = 0$, *then* $C(X_k)e_kC(X_k)$ *are heredity ideals and the ideal* $C(Y)\chi_Z C(Y)$ *is strong idempotent.*

Proof. We shall prove corollary by induction on m. For $m = 1$ the claim follows from Theorem 1.31.

Suppose we proved corollary for all $m \leq N-1$. We shall prove it for $m = N$. Let us check that we can apply Theorem 1.31 to the sets $X' = X \cup \{z_1, \dots, z_{N-1}\}$ and $Z' = \{z_N\}$. We have $\Gamma_2 Z' \cap Y = Z'$. Suppose that this does not hold, then there exists $\gamma \in \Gamma$ such that $\gamma z_N \in Y$ and $\gamma z_N \neq z_N$. Then, since $\Gamma_2 Z \cap Y = Z$, we have $\gamma z_N = z_j$ for some $j < N$. But $z_N <_{\Gamma_2} \gamma z_N = z_j$, that should imply $N < j$. Contradiction. Thus $\Gamma_2 Z' \cap Y = Z'$.

Further $\Gamma_1 X' \cap Y = X'$. In fact, suppose there is $y \in X'$ and $\gamma \in \Gamma_1$ such that $\gamma y = z_N$. Since $\Gamma_1 X \cap Y = X$, we have $y \in Z$, that is $y = z_j$ for some $j \leq N - 1$. But then $z_j = y <_{\Gamma_1} \gamma y = z_N$ and therefore $N < j$. Contradiction.

From Theorem 1.31 we get

(i) $C(X_N)e_NC(X_N) = C(Y)e_NC(Y)$ is a 2-idempotent ideal;

(ii) the algebra $C(X_{N-1}) = C(X')$ has a basis

$$J = \{a_1 a_2 \otimes \chi_x | a_1 \in J_1,\ a_2 \in I_x; x, \deg(a_2)x, \deg(a_1 a_2)x \in X_{N-1}\}.$$

For all $x \in X_{N-1}$ denote by ${I_x}'$ the set

$$\{a | a \in I_x, \deg(a)x \neq z_N\}.$$

Then

$$J = \left\{a_1 a_2 \otimes \chi_x \middle| a_1 \in J_1, a_2 \in {I_x}'; x, \deg(a_1 a_2)x \in X_{N-1}\right\}.$$

We have $C_2(Y)e_NC_2(Y) = e_NC_2(Y)$. Therefore

$$\{a\chi_x | a \in I_x; x, \deg(a)x \neq z_N\} = \left\{a\chi_x \middle| a \in {I_x}',\ x \in X_{N-1}\right\}.$$

is a basis of the algebra $C_2(X_{N-1})$. Denote by Z'' the set $\{z_1, \dots, z_{N-1}\}$. Since

$$\Gamma_2 Z'' \cap X_{N-1} = \Gamma_2 Z'' \cap Y \cap X_{N-1} = Z \cap X_{N-1} = Z'',$$

we can apply the claim of corollary to the set $X_{N-1} = X \coprod Z''$ with $m = N-1$.

Now, suppose that $\operatorname{Rad}(A_\epsilon) = 0$, then

$$e_k \operatorname{Rad}(C(X_k))e_k = \operatorname{Rad}(e_k C(X_k)e_k) = \operatorname{Rad}(e_k C e_k) = \operatorname{Rad}(A_\epsilon) = 0,$$

since $e_k C e_k = \chi_{z_k} C \chi_{z_k} \cong A_\epsilon$. Now the claim follows from Corollary 1.27 and Proposition 1.21. ☐

Theorem 1.35. *Let Γ be a* commutative *positive monoid. Suppose that the Γ-graded algebra A can be decomposed as the tensor product $A_1 \otimes A_2 \otimes \dots A_m$, such that for all $1 \le i < j \le m$*

$$A_{ij} = A_i \otimes A_{i+1} \otimes \cdots \otimes A_j$$

is a Γ-graded subalgebra of A.

Let J_i be a Γ-homogeneous basis of A_i for $1 \le i \le m$, and $Y = \coprod_{i=1}^m Z_i$ a Γ-convex subset of S. Denote by Y_j the set $Z_1 \coprod \cdots \coprod Z_j$ and by e_j the idempotent $\chi_{\bar{Y}_j}$.

If there are submonoids Γ_i of Γ such that

(i) the subalgebra A_i is Γ_i-graded, that is, for all $\gamma \in \Gamma \setminus \Gamma_i$ the space $(A_i)_\gamma$ is zero;

(ii) $\Gamma_j Z_i \cap Y \subset \coprod_{k=i}^{j} Z_k$ for $1 \le i \le j \le m$;

(iii) $\Gamma_i Y_j \cap Y = Y_j$ for $1 \le i < j \le m$;

then for each $1 \le j \le m-1$ the natural map

$$C(Y_{j+1})e_j \otimes_{e_j C(Y_{j+1}) e_j} e_j C(Y_{j+1}) \to C(Y_{j+1}) e_j C(Y_{j+1})$$

is an isomorphism of vector spaces.

Moreover, for $1 \le j \le m$ the set

$$\left\{ a_1 a_2 \dots a_m \chi_y \,\middle|\, \begin{array}{l} a_k \in J_k,\ 1 \le k \le m; y \in Y_j \\ \deg(a_k a_{k+1} \dots a_m) y \in Y_j,\ j \le k \le m \end{array} \right\}$$

is a basis of $C(Y_j)$. Suppose additionally that $\operatorname{Rad}(A_\epsilon) = 0$ and that on each set Z_j, $j \ge 2$ there is an ordering $\le_j$, satisfying

(i) $z \le_j z'$ if $z \le_{\Gamma_i} z'$, for all $i \ge j$;

(ii) $z \ge_j z'$ if $z \le_{\Gamma_i} z'$, for all $i < j$.

Then the ideals $C(Y_j)e_{j-1}C(Y_j)$, for $j \geq 2$ are strong idempotent.

By Proposition 1.21 we have also that the ideal $C(Y)\chi_{\bar{Z}_1}C(Y)$ of $C(Y)$ is strong idempotent.

Proof. We prove the theorem by induction on n. The case $m = 1$ is proved in Corollary 1.30.

Suppose the claim of the theorem holds for all $m \leq N$. We then prove it for $m = N + 1$.

Decompose Y as the disjoint union of N sets Y_2, Z_3,..., Z_{N+1}. Denote by Γ_{12} the submonoid of Γ generated by Γ_1 and Γ_2. Then A_{12} is a Γ_{12}-graded algebra. We claim that the conditions of the theorem are satisfied for the same set Y and the same algebra A and

(i) $m = N$;

(ii) $Z'_1 = Y_2$, $Z'_2 = Z_3$, $\dots$, $Z'_N = Z_{N+1}$;

(iii) $A'_1 = A_{12}$, $A'_2 = A_3$,$\dots$, $A'_N = A_{N+1}$;

(iv) $\Gamma'_1 = \Gamma_{12}$, $\Gamma'_2 = \Gamma_3$, $\dots$, $\Gamma'_N = \Gamma_{N+1}$.

In fact

$$A'_{ij} = A'_i \otimes A'_{i+1} \otimes \cdots \otimes A'_j = \begin{cases} A_{1,j+1} & \text{if } i = 1 \\ A_{i+1,j+1} & \text{if } i \neq 1 \end{cases}$$

are subalgebras of A by the hypothesis of the theorem. Now, for $j \geq i$ and $i \neq 1$

$$\Gamma'_j Z'_i \cap Y = \Gamma_{j+1} Z_{i+1} \cap Y \subset \coprod_{k=i+1}^{j+1} Z_k = \coprod_{k=i}^{j} Z'_k,$$

$$\Gamma'_i Y'_j \cap Y = \Gamma_{i+1} Y_{j+1} \cap Y = Y_{j+1} = Y'_j.$$

For $i = 1$ and $j > 1$ we have $j + 1 \geq 2$ and therefore

$$\begin{aligned} \Gamma'_j Z'_1 \cap Y &= \Gamma_{j+1} Y_2 \cap Y = \Gamma_{j+1}(Z_1 \sqcup Z_2) \cap Y \\ &\subset (\Gamma_{j+1} Z_1 \cap Y) \cup (\Gamma_{j+1} Z_2 \cap Y) \\ &\subset \coprod_{k=1}^{j+1} Z_k \cup \coprod_{k=2}^{j+1} Z_k = \coprod_{k=1}^{j} Z'_k. \end{aligned}$$

Note, that $\Gamma'_1 Z'_1 \cap Y \subset Z'_1$ is equivalent to $\Gamma'_1 Y'_1 \cap Y \subset Y'_1$. Thus it is only needed to check that for $i = 1$ and $j \geq 1$ we have $\Gamma'_1 Y'_j \cap Y = Y'_j$. Or in other words, that $\Gamma_{12} Y_{j+1} \cap Y = Y_{j+1}$. Note, that $\Gamma_1 Y_{j+1} \cap Y = Y_{j+1}$ and $\Gamma_2 Y_{j+1} \cap Y = Y_{j+1}$ by the condition of the theorem. Now, we use Γ-convexity of Y. Let $\gamma \in \Gamma_{12}$. Then γ can be written as a product $\gamma_1\gamma_2$ with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Let $y \in Y_{j+1}$ be such that $\gamma y \in Y$. Then we have

$$y \leq_\Gamma \gamma_2 y \leq_\Gamma \gamma_1\gamma_2 y = \gamma y.$$

Since both y and γy are elements of Y and Y is Γ-convex, the element $\gamma_2 y$ lies in Y. Now

$$\gamma_2 y \in \Gamma_2 Y_{j+1} \cap Y = Y_{j+1},$$
$$\gamma y = \gamma_1(\gamma_2 y) \in \Gamma_1 Y_{j+1} \cap Y = Y_{j+1}.$$

Thus $\Gamma_{12} Y_{j+1} = Y_{j+1}$.

Therefore, from the induction assumption for $2 \le j \le N$ the natural map

$$C(Y_{j+1})e_j \otimes_{e_j C(Y_{j+1}) e_j} e_j C(Y_{j+1}) \to C(Y_{j+1}) e_j C(Y_{j+1})$$

is an isomorphism. If the additional ordering assumptions are satisfied for sets $Z_2,\ Z_3,\ \dots, Z_{N+1}$ and monoids $\Gamma_2,\ \Gamma_3, \dots,\ \Gamma_{N+1}$ then they are automatically satisfied for sets $Z_2' = Z_3,\ Z_3' = Z_4,\ \dots, Z_N' = Z_{N+1}$ and monoids $\Gamma_2' = \Gamma_3$, $\Gamma_3' = \Gamma_4,\ \dots,\ \Gamma_N' = \Gamma_{N+1}$. Therefore, in that case, the ideals $C(Y_j)e_{j-1}C(Y_j)$ for $3 \le j \le N$ are strong idempotent.

Returning to the general case, we now explore the consequences of the induction hypothesis on bases. Since $\{a_1 a_2 | a_1 \in J_1, a_2 \in J_2\}$ is a homogeneous basis of A_{12}, the sets

$$\left\{ a_1 a_2 \dots a_{N+1} \chi_y \;\middle|\; \begin{array}{l} a_k \in J_k,\ 1 \le k \le N+1 \\ \deg(a_k a_{k+1} \dots a_{N+1}) y \in Y_j,\ j \le k \le N+1 \\ y \in Y_j \end{array} \right\}$$

are bases of $C(Y_j)$ for $3 \le j \le N+1$, and the set

$$\left\{ (a_1 a_2) a_3 \dots a_{N+1} \chi_y \;\middle|\; \begin{array}{l} a_k \in J_k,\ 1 \le k \le N+1 \\ y \in Y_2 \\ \deg(a_k a_{k+1} \dots a_{N+1}) y \in Y_2,\ 3 \le k \le N+1 \\ \deg((a_1 a_2) a_3 \dots a_{N+1}) y \in Y_2 \end{array} \right\}$$

is a basis of $C(Y_2)$.

Let $(a_1 a_2) a_3 \dots a_{N+1} \chi_y$ be an element of the last set. Denote by z the element $\deg(a_3 \dots a_{N+1}) y$. Then $z \in Y_2$. We have

$$z < \deg(a_2) z < \deg(a_1) \deg(a_2) z = \deg(a_1 a_2) z.$$

Since z and $\deg(a_1 a_2) z$ are elements of Y_2, which is a subset of Y, and Y is Γ-convex, it follows that $\deg(a_2) z$ is an element of Y. Now

$$\deg(a_2) z \in \Gamma_2 Y_2 \cap Y = (\Gamma_2 Z_1 \cap Y) \cup (\Gamma_2 Z_2 \cap Y) \subset (Z_1 \sqcup Z_2) \sqcup Z_2 = Y_2.$$

Therefore the above basis of $C(Y_2)$ can be written as

$$\left\{ (a_1 a_2) a_3 \dots a_{N+1} \chi_y \;\middle|\; \begin{array}{l} a_k \in J_k,\ 1 \le k \le N+1 \\ y \in Y_2 \\ \deg(a_k a_{k+1} \dots a_{N+1}) y \in Y_2,\ 1 \le k \le N+1 \end{array} \right\}.$$

Now, the algebra $A_{2,N}$ with the decomposition $A_2 \otimes A_3 \otimes \dots A_{N+1}$, submonoids Γ_j, $2 \le j \le N+1$ and decomposition $Y_2 \coprod Z_3 \coprod \dots \coprod Z_{N+1}$ of Y

satisfy the conditions of the theorem for $m = N$. By the induction hypothesis we get that the set

$$\left\{ a_2 a_3 \ldots a_{N+1}\chi_y \,\middle|\, \begin{array}{l} a_k \in J_k,\ 2 \le k \le N+1 \\ y \in Y_2 \\ \deg(a_k a_{k+1} \ldots a_{N+1})y \in Y_2,\ 2 \le k \le N+1 \end{array} \right\}$$

is a basis of $C_{2,N+1}(Y_2) = (A_{2,N+1} \ltimes_\Gamma B)(Y_2)$.

For each $y \in Y_2$ denote by I_y the set

$$\left\{ a_2 a_3 \ldots a_{N+1} \,\middle|\, \begin{array}{l} a_k \in J_k,\ 2 \le k \le N+1 \\ \deg(a_k a_{k+1} \ldots a_{N+1})y \in Y_2,\ 2 \le k \le N+1 \end{array} \right\}.$$

Then the set

$$\left\{ a_1 \tilde{a}\chi_y \,\middle|\, \begin{array}{l} a_1 \in J_1; \tilde{a} \in I_y \\ y \in Y_2; \deg(a_1\tilde{a})y \in Y_2 \end{array} \right\}$$

is a basis of $C(Y_2)$ and the set

$$\left\{ \tilde{a}\chi_y \,\middle|\, \tilde{a} \in I_y; y \in Y_2 \right\}$$

is a basis of $C_{2,N+1}(Y_2)$.

Denote by $\Gamma_{2,N+1}$ the submonoid of Γ generated by Γ_2, ..., Γ_{N+1}. Then $A_{2,N+1}$ is $\Gamma_{2,N+1}$-graded algebra. We have

$$\Gamma_1 Z_1 \cap Y_2 \subset \Gamma_1 Z_1 \cap Y = Z_1.$$

Next we show that $\Gamma_{2,N+1} Z_2 \cap Y_2 = Z_2$. Let $\gamma \in \Gamma_{2,N+1}$ and $z \in Z_2$ be such that $\gamma z \in Y_2$. Since Γ is commutative, we can write γ as a product $\gamma_{N+1} \ldots \gamma_2$, where each γ_s belongs to some Z_s. Then we have

$$z \le_\Gamma \gamma_2 z \le_\Gamma \gamma_3\gamma_2 z \le_\Gamma \cdots \le_\Gamma \gamma z.$$

Since both elements z and γz lie in Y and Y is Γ-convex it follows, that element $\gamma_s \ldots \gamma_2 z$ belongs to Y. Now

$$\gamma_2 z \in \Gamma_2 Z_2 \cap Y = Z_2,$$

$$\gamma_3\gamma_2 z \in \Gamma_3 Z_2 \cap Y \subset Z_2 \sqcup Z_3,$$

$$\gamma_4\gamma_3\gamma_2 z \in \Gamma_4(Z_2 \sqcup Z_3) \cap Y \subset Z_2 \sqcup Z_3 \sqcup Z_4.$$

Proceeding this way, we get

$$\gamma_s \ldots \gamma_2 z \in \coprod_{k=2}^{s} Z_k.$$

In particular

$$\gamma z \in \coprod_{k=2}^{N+1} Z_k.$$

But, since $z \in Y_2 = Z_1 \sqcup Z_2$, this means that $\gamma z \in Z_2$. Thus

$$\Gamma_{2,N+1} Z_2 \cap Y_2 = Z_2.$$

Therefore, we can apply Theorem 1.31 to the decomposition $A = A_1 \otimes A_{2,N+1}$ of A and $Y_2 = Z_1 \sqcup Z_2$. Note that $e_1 = \chi_{Z_2} \in C(Y_2)$. Therefore

$$C(Y_2)e_1 \otimes_{e_1 C(Y_2) e_1} C(Y_2) \to C(Y_2)e_1 C(Y_2)$$

is an isomorphism. Moreover, if $\mathrm{Rad}(A_\epsilon) = 0$, and there is an ordering $\leq_2$ on Z_2 such that

(i) $z \leq_{\Gamma_i} z' \Rightarrow z \leq_2 z'$, for all $i \geq 2$;

(ii) $z \leq_{\Gamma_1} z' \Rightarrow z \geq_2 z'$,

then

$$z \leq_{\Gamma_{2,N}} z' \Rightarrow z \leq_2 z'.$$

Hence , we can apply Corollary 1.34 and get that the ideal $C(Y_2)e_1C(Y_2)$ is strong idempotent.

Now, returning to the general case, the set

$$\{a_1 \tilde{a} \chi_y | a_1 \in J_1; \tilde{a} \in I_y; y, \deg(\tilde{a})y, \deg(a_1 \tilde{a})y \in Y_1\} =$$

$$= \left\{ a_1 a_2 \dots a_{N+1} \chi_y \;\middle|\; \begin{array}{l} a_k \in J_k,\ 1 \leq k \leq N+1 \\ y, \deg(a_2 \dots a_{N+1})y, \deg(a_1 \dots a_{N+1})y \in Y_1 \\ \deg(a_k a_{k+1} \dots a_{N+1})y \in Y_2,\ 3 \leq k \leq N+1 \end{array} \right\}$$

is a basis of $C(Z_1) = C(Y_1)$. Let $a_1 \dots a_{N+1}\chi_y$ be an element of the last set. We know, that

$$\deg(a_k \dots a_{N+1})y \in Y_2, \text{ for } 3 \leq k \leq N+1.$$

Assume, that

$$\deg(a_k \dots a_{N+1})y \in Z_2,$$

for some k. Then

$$\deg(a_2 \dots a_{N+1})y = \deg(a_2 \dots a_{k-1}) \deg(a_k \dots a_{N+1})y \in Z_1 \cap (\Gamma_{2,N+1} Z_2).$$

But

$$Z_1 \cap (\Gamma_{2,N+1} Z_2) = Z_1 \cap Y_2 \cap (\Gamma_{2,N+1} Z_2) = Z_1 \cap Z_2 = \phi.$$

Thus,

$$\deg(a_k \dots a_{N+1})y \in Z_1, \text{ for all } k$$

and the set

$$\left\{ a_1 a_2 \dots a_{N+1} \chi_y \;\middle|\; \begin{array}{l} a_k \in J_k,\ 1 \leq k \leq N+1 \\ y \in Y_1 \\ \deg(a_k a_{k+1} \dots a_{N+1})y \in Y_1,\ 1 \leq k \leq N+1 \end{array} \right\}$$

is a basis of the algebra $C(Y_1)$. $\square$

2 Application to Schur algebras

In this section we apply the technique developed in the previous section to the problem of constructing (minimal) projective resolutions for simple modules over the Borel subalgebra $S^+(n,r)$ of the Schur algebra $S(n,r)$. We start with a short overview of Schur algebras.

2.1 Results on combinatorics

We shall use the following combinatorial notions.

Definition 2.1. A *partition* λ of r is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative weakly decreasing integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $\sum \lambda_i = r$. The set of all partitions of r is denoted by $\Lambda^+(r)$. The λ_i are the *parts* of the partition. If $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$, we say λ has *length* at most n. The set of all partitions of length at most n is denoted by $\Lambda^+(n,r)$.

Dropping the condition that the λ_i are decreasing, we say that λ is a *composition* of r. The set of all compositions of r is denoted by $\Lambda(r)$. The set of all compositions of r of length at most n is denoted by $\Lambda(n,r)$.

There is a natural ordering on the set $\Lambda(r)$:

Definition 2.2. (Dominance order) For $\lambda, \mu \in \Lambda(r)$, we say that λ *dominates* μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all j.

Restricting, we get the dominance order on $\Lambda(n,r)$. Now, let $\mathbb{Z}^n$ be the n-dimensional lattice over the ring of integer numbers. We can consider $\Lambda(n,r)$ as a subset $\mathbb{Z}^n$. We extend the dominance order from $\Lambda(n,r)$ to $\mathbb{Z}^n$ by the following definition

Definition 2.3. (Dominance order) For $z, \bar{z} \in \mathbb{Z}^n$, we say that $\bar{z}$ *dominates* z and write $\bar{z} \trianglerighteq z$ if

$$\sum_{i=1}^{j} \bar{z}_i \geq \sum_{i=1}^{j} z_i$$

for all j.

Let Ψ_n be the submonoid of $\mathbb{Z}^n$ generated by the vectors $v_i - v_{i+1}$, where $\{v_i \mid 1 \leq i \leq n\}$ is the standard basis of $\mathbb{Z}^n$. Then Ψ_n acts effectively on $\mathbb{Z}^n$ by bijections if we define $\gamma z := \gamma + z$. The partial order $\leq_{\Psi_n}$ introduced in Section 1.4, now becomes

$$z \leq_{\Psi_n} \bar{z} \text{ iff } \bar{z} - z \in \Psi_n.$$

Proposition 2.4. *The dominance order on $\mathbb{Z}^n$ coincides with $<_{\Psi_n}$.*

Proof. It is clear that for any $z \in \mathbb{Z}^n$ and any i between 1 and $n-1$:

$$z + v_i - v_{i+1} \trianglerighteq z.$$

Thus $\bar{z} >_{\Psi_n} z$ implies $\bar{z} \trianglerighteq z$.

Let z and $\bar{z} \in \mathbb{Z}^n$ be such that $\bar{z} \trianglerighteq z$. Denote by k_j the difference

$$\sum_{i=1}^{j} \bar{z}_i - \sum_{i=1}^{j} z_i.$$

Then for all j the number k_j is positive. It easy to see, that

$$\bar{z} - z = \sum_{j=1}^{n-1} k_j (v_j - v_{j+1}) \in \Psi_n.$$

Thus $\bar{z} >_{\Psi_n} z$. ◻

It is clear that $(r, 0, \ldots, 0)$ and $(0, \ldots, 0, r)$ are the maximal and minimal elements of $\Lambda(n, r)$ with respect to the dominance order, respectively. We denote by $\Lambda^1(n, r)$ the smallest Ψ_n-convex subset in $\mathbb{Z}^n$ containing $(r, 0, \ldots, 0)$ and $(0, \ldots, 0, r)$. Then $\Lambda^1(n, r)$ contains $\Lambda(n, r)$ but does not coincide with it, since $z \in \Lambda^1(n, r)$ can have negative coordinates.

Proposition 2.5. *For all $z \in \Lambda^1(n, r)$ we have $z_1 \geq 0$ and*

$$\sum_{i=1}^{n} z_i = r.$$

Proof. Since z dominates $(0, \ldots, 0, r)$ we have

$$\sum_{i=1}^{j} z_i \geq 0$$

for all $j \in \{1, 2, \ldots, n-1\}$. In particular, $z_1 \geq 0$. Moreover,

$$\sum_{i=1}^{n} z_i \geq r.$$

Since z is dominated by $(r, 0, \ldots, 0)$ we have

$$\sum_{i=1}^{n} z_i \leq r.$$

Thus the claim of proposition is proved. ◻

We denote by $\Lambda^k(n,r)$ the subset of $\Lambda^1(n,r)$ of all z such that $z_i \geq 0$ for all $i \in \{1,2,\ldots,k\}$. Then $\Lambda^l(n,r) \subset \Lambda^k(n,r)$ if $l \geq k$ and $\Lambda^n(n,r) = \Lambda(n,r)$. Denote by $M^k(n,r)$ the difference $\Lambda^{k-1}(n,r) \setminus \Lambda^k(n,r)$. Then

$$M^k(n,r) = \left\{(z_1,\ldots,z_n)\middle| z_1 \geq 0,\ldots,z_{k-1} \geq 0, z_k < 0,\, z \in \Lambda^1(n,r)\right\}.$$

Denote by $\Psi_{n,k}$ the submonoid of Ψ_n generated by the elements $v_i - v_k$ for all $i \in \{1,2,\ldots,k-1\}$. Note, that $\Psi_{n,k} \cap \Psi_{n,l} = \{0\}$, if $k \neq l$. Further, let

$$\Phi_{n,k} = \Psi_{n,2} + \Psi_{n,3} + \cdots + \Psi_{n,k}$$

and

$$\Theta_{n,k} = \Psi_{n,k+1} + \Psi_{n,k+2} + \cdots + \Psi_{n,n}.$$

Note, that $\Phi_{n,k} \cap \Theta_{n,k} = \{0\}$ for all $k \in \{2,3,\ldots,n\}$.

Next we will introduce an order $\leq_k$ on $M^k(n,r)$, satisfying

(i) $z_1 \leq_{\Phi_{n,k}} z_2 \Rightarrow z_1 \leq_k z_2$;

(ii) $z_1 \leq_{\Theta_{n,k}} z_2 \Rightarrow z_1 \geq_k z_2$

for $z_1,\ z_2 \in M^k(n,r)$. For this let $\leq_{lex}$ denote the lexicographic order on $\mathbb{Z}^n$. Define the map

$$\begin{aligned} \phi^k\colon \mathbb{Z}^n &\to \mathbb{Z}^{n+1} \\ (z_1,z_2,\ldots,z_n) &\mapsto \left(\sum_{i\neq k} z_i,\ \sum_{i=k+1}^{n} z_i, z_1, z_2,\ldots,z_{k-1}, z_n, z_{n-1},\ldots,z_{k+1}\right). \end{aligned}$$

Define the order $\leq_k$ on $M^k(n,r)$ by

$$z \leq_k z' \equiv \phi^k(z) \leq_{lex} \phi^k(z').$$

Proposition 2.6. *The order $\leq_k$ satisfies the above stated properties:*

(i) $z_1 \leq_{\Phi_{n,k}} z_2 \Rightarrow z_1 \leq_k z_2$*;*

(ii) $z_1 \leq_{\Theta_{n,k}} z_2 \Rightarrow z_1 \geq_k z_2$*.*

Proof. Note, that $\Phi_{n,k}$ is generated by the vectors $v_i - v_j$, with $i < j \leq k$. To prove the first property it is enough to check that for all $z \in M^k(n,r)$

$$\phi^k(z) \leq_{lex} \phi^k(z + v_i - v_j),\ \text{if } i < j \leq k.$$

But ϕ^k is a linear map and $\leq_{lex}$ is compatible with addition. Thus, it is enough to check that $\phi^k(v_i - v_j) \geq_{lex} 0$. For $j < k$ we have

$$\phi^k(v_i - v_j) = (0,0,0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0) \geq_{lex} 0.$$

Further

$$\phi^k(v_i - v_k) = (1,\ldots) \geq_{lex} 0.$$

The monoid $\Theta_{n,k}$ is generated by the vectors $v_i - v_j$ with $i < j$ and $k+1 \leq j \leq n$. To prove the second part of the proposition it is enough to check that

$$\phi^k(v_i - v_j) \leq_{lex} 0.$$

For $i < k < j$ we have

$$\phi^k(v_i - v_j) = (0, -1, \dots) \leq_{lex} 0.$$

For $k < i < j$ we get

$$\phi^k(v_i - v_j) = (0, 0, 0, \dots, 0, 0, \dots, -1, \dots) \leq_{lex} 0.$$

Finally

$$\phi^k(v_k - v_j) = (-1, \dots) \leq_{lex} 0.$$

$\square$

Denote by $I(n,r)$ the subset of $\mathbb{N}^r$ consisting from the elements $i = (i_1, i_2, \dots, i_r)$, such that $i_k \in \{1, 2, \dots, n\}$ for all k between 1 and r.

Definition 2.7. We write $i \leq j$ for i, $j \in I(n,r)$ if $i_\sigma \leq j_\sigma$ for all σ, $1 \leq \sigma \leq r$.

Denote by Σ_r the permutation group on $\{1, 2, \dots, r\}$. The group Σ_r acts on $I(n,r)$ by the rule

$$i\pi = (i_{\pi(1)}, \dots, i_{\pi(r)}) \qquad (i \in I,\ \pi \in \Sigma_r).$$

Then we can extend the action of Σ_r on $I(n,r)^2$ by

$$(i,j)\pi = (i\pi, j\pi).$$

We denote by $\Omega(n,r)$ the set

$$\{(i,j) \in I(n,r) \times I(n,r) \,|\, i \leq j\} \Big/ \Sigma_r\ .$$

Definition 2.8. We say that a composition $\lambda = (\lambda_1, \dots, \lambda_n)$ is the *weight* of $i \in I(n,r)$, written $\lambda = \mathrm{wt}(i)$, if

$$\lambda_\nu = |\{\rho \in \{1, 2, \dots, r\} | i_\rho = \nu\}|$$

for all $\nu \in \{1, 2, \dots, n\}$.

It is clear that if $i \leq j$, then $\mathrm{wt}(i) \trianglerighteq \mathrm{wt}(j)$. For λ,$\mu \in \Lambda(n,r)$ such that $\lambda \trianglerighteq \mu$ let

$$\Omega(\lambda, \mu) = \{(i,j) \in I(n,r) \times I(n,r) \,|\, i \leq j,\ \mathrm{wt}(i) = \lambda,\ \mathrm{wt}(j) = \mu\} \Big/ \Sigma_r\ .$$

We denote by $T(n,r)$ the set of upper-triangular $n \times n$-matrices over $\mathbb{N}$ such that the sum of all its entries is r. Let

$$T(\lambda, \mu) = \left\{ K = (k_{\sigma\rho})_{\sigma,\rho=1}^n \middle| \sum_{\rho=\sigma}^n k_{\sigma\rho} = \lambda_\sigma,\ \sum_{\sigma=1}^{\rho} k_{\sigma\rho} = \mu_\rho,\ \sigma, \rho \in \{1, 2, \dots, n\} \right\}.$$

Proposition 2.9. *For i,$j \in I(n,r)$ such that $i \leq j$ and σ,$\rho \in \{1,2,\ldots,n\}$, set*

$$t(i,j)_{\sigma,\rho} = \#\left\{\tau \in \{1,2,\ldots,r\} | i_\tau = \sigma,\, j_\tau = \rho\right\}.$$

Then the map

$$\begin{aligned} t\colon \{(i,j) \in I(n,r) \times I(n,r) |\, i \leq j,\, \mathrm{wt}(i) = \lambda,\, \mathrm{wt}(j) = \mu\} \to T(\lambda,\mu) \\ (i,j) \mapsto \left(t(i,j)_{\sigma\rho}\right)_{\sigma,\rho=1}^{n} \end{aligned}$$

induces a bijection between $\Omega(\lambda,\mu)$ and $T(\lambda,\mu)$.

Proof. Since $i \leq j$, we get that for $\sigma > \rho$ the number $t(i,j)_{\sigma,\rho} = 0$. Moreover

$$\begin{aligned} \sum_{\rho=\sigma}^{n} t(i,j)_{\sigma\rho} &= \sum_{\rho=\sigma}^{n} \#\left\{\tau \in \{1,2,\ldots,r\} | i_\tau = \sigma,\, j_\tau = \rho\right\} \\ &= \#\left\{\tau \in \{1,2,\ldots,r\} | i_\tau = \sigma\right\} \\ &= \mathrm{wt}(i)_\sigma = \lambda_\sigma \end{aligned}$$

and

$$\begin{aligned} \sum_{\sigma=1}^{\rho} t(i,j)_{\sigma\rho} &= \sum_{\sigma=1}^{\rho} \#\left\{\tau \in \{1,2,\ldots,r\} | i_\tau = \sigma,\, j_\tau = \rho\right\} \\ &= \#\left\{\tau \in \{1,2,\ldots,r\} |\, j_\tau = \rho\right\} \\ &= \mathrm{wt}(j)_\rho = \mu_\rho. \end{aligned}$$

Thus the image of t lies in $T(\lambda,\mu)$. It is clear that t is Σ_r-invariant. Thus t induces a map from $\Omega(\lambda,\mu)$ to $T(\lambda,\mu)$. For $K = (k_{\sigma,\rho}) \in T(\lambda,\mu)$ we set

$$i = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n})$$

and

$$j = (1^{k_{11}}, 2^{k_{12}}, \ldots, n^{k_{1n}}, 2^{k_{22}}, \ldots, n^{k_{nn}}).$$

Then $i \leq j$ and $t(i,j)_{\sigma,\rho} = k_{\sigma,\rho}$. Thus t is surjective. Moreover, each pair (i',j') of multi-indices, such that $t(i',j') = K$, can be transformed to (i,j) with the appropriate element of Σ_r. Thus t is injective. ☐

Corollary 2.10. *There is a bijection between $\Omega(n,r)$ and $T(n,r)$.*

Proof. In fact, $\Omega(n,r) = \coprod_{\lambda \trianglerighteq \mu} \Omega(\lambda,\mu)$ and $T(n,r) = \coprod_{\lambda \trianglerighteq \mu} T(\lambda,\mu)$. By Proposition 2.9 there is a bijection between sets $\Omega(\lambda,\mu)$ and $T(\lambda,\mu)$ for all $\lambda \trianglerighteq \mu$. ☐

2.2 The Schur algebra $S(n,r)$ and the Borel subalgebra $S^+(n,r)$

In this section we follow [10, 14, 15].

Let $\mathbb{K}$ be an infinite field (of any characteristic) and V the natural module over $\mathrm{GL}_n(\mathbb{K})$ with basis $\{v_1,\ldots,v_n\}$. Then there is a diagonal action of $\mathrm{GL}_n(\mathbb{K})$

on the r-fold tensor product $V^{\otimes r}$. With respect to the basis $\{v_i = v_{i_1} \otimes \cdots \otimes v_{i_r} : i \in I(n,r)\}$, this action is given by the formula

$$gv_i = gv_{i_1} \otimes \cdots \otimes gv_{i_r}.$$

We denote by $\tau_{n,r}\colon \mathrm{GL}_n(\mathbb{K}) = \mathrm{GL}(V) \to \mathrm{End}_{\mathbb{K}}(V^{\otimes r})$ the corresponding representation of the group $\mathrm{GL}_n(\mathbb{K}) = \mathrm{GL}(V)$.

Definition 2.11 ([10])**.** The Schur algebra $S_{\mathbb{K}}(n,r)$ is the linear closure of the group $\{\tau_{n,r}(g) : g \in \mathrm{GL}_n(\mathbb{K})\}$.

Let us denote by $B_n^+(\mathbb{K})$ the subgroup of $\mathrm{GL}_n(\mathbb{K})$ consisting of the upper triangular matrices.

Definition 2.12 ([11])**.** The upper Borel subalgebra $S_{\mathbb{K}}^+(n,r)$ of the Schur algebra $S_{\mathbb{K}}(n,r)$ is the linear closure of the group $\{\tau_{n,r}(g) | g \in B_n^+(\mathbb{K})\}$.

We denote by $e_{i,j}$ the linear transformation of $V^{\otimes r}$ whose matrix, relative to the basis $\{\, v_i : i \in I(n,r) \,\}$ of $V^{\otimes r}$, has 1 in place (i,j) and zeros elsewhere.

Define

$$\xi_{i,j} = \sum_{\pi \in \Sigma_r} e_{i\pi, j\pi}.$$

Proposition 2.13 ([14, Thm. 2.2.6])**.** *The set*

$$\left\{ \xi_{i,j} \,\middle|\, \overline{(i,j)} \in \left(I(n,r) \times I(n,r) \middle/ \Sigma_r \right) \right\}$$

is a $\mathbb{K}$*-basis of* $S(n,r)$*.*

The next statement was proved in [11, §§3, 6].

Proposition 2.14. *The algebra* $S_{\mathbb{K}}^+(n,r)$ *has a* $\mathbb{K}$*-basis* $\left\{ \xi_{i,j} : \overline{(i,j)} \in \Omega(n,r) \right\}$.

2.3 Universal enveloping algebra of sl_n^+ and Kostant form

Denote by sl_n^+ the lie algebra of upper triangular nilpotent matrices. Let $\mathfrak{A}_n(\mathbb{C})$ be its universal enveloping algebra over $\mathbb{C}$. We shall consider sl_n^+ with the standard basis $\{e_{ij} \mid 1 \le i < j \le n\}$. Then $\mathfrak{A}_n(\mathbb{C})$ is generated as an algebra by the elements $e_{1,2}, e_{2,3}, \ldots, e_{n-1,n}$.

We order the elements e_{ij} in such way, that $e_{ij} \le e_{i'j'}$ if and only if

$$(j,i) \ge_{lex} (j',i').$$

In other words,

$$e_{12} > e_{13} > e_{23} > \cdots > e_{1k} > e_{2k} > \cdots > e_{k-1,k} > \cdots > e_{n-1,n}.$$

We always assume that in the product $\prod_{i<j} e_{ij}^{k_{ij}}$ the generators increase from the left to right, with respect to the above order. For example, if $n = 3$ and

$$(k_{ij})_{i,j=1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\prod_{i<j} e_{ij}^{k_{ij}} = e_{23}^2 e_{13} e_{12}.$$

It follows from the Poincare-Birkhoff-Witt Theorem, that the set

$$\mathbb{B}_n = \left\{ \prod_{1 \le i < j \le n} e_{ij}^{k_{ij}} \middle| k_{ij} \in \mathbb{N} \right\}$$

is a $\mathbb{C}$-basis of $\mathfrak{A}_n(\mathbb{C})$. Denote by $e_{ij}^{(k)}$ the element $\frac{1}{k!} e_{ij}^k$ of the algebra $\mathfrak{A}_n(\mathbb{C})$. We define $\mathfrak{A}_n(\mathbb{Z})$ to be the Z-sublattice of $\mathfrak{A}_n(\mathbb{C})$ generated by the set

$$\overline{\mathbb{B}}_n = \left\{ \prod_{i<j} e_{ij}^{(k_{ij})} \middle| k_{ij} \in \mathbb{N} \right\}.$$

Proposition 2.15. *The set $\mathfrak{A}_n(\mathbb{Z})$ is a subring of $\mathfrak{A}_n(\mathbb{C})$. In other words, $\mathfrak{A}_n(\mathbb{Z})$ is a $\mathbb{Z}$-algebra. It is called the* Kostant form *of the universal enveloping algebra $\mathfrak{A}_n(\mathbb{C})$ over $\mathbb{Z}$.*

Proof. For a proof see [12, Lemma 2 after Proposition 3] and [12, Remark 3] thereafter. □

Definition 2.16. For any field $\mathbb{K}$, the algebra $\mathfrak{A}_n(\mathbb{K}) := \mathbb{K} \otimes_{\mathbb{Z}} \mathfrak{A}_n(\mathbb{Z})$ is called *Kostant form* of the algebra $\mathfrak{A}_n(\mathbb{C})$ over $\mathbb{K}$.

Define a degree function on $\overline{\mathbb{B}}_n$, by

$$\begin{aligned} \deg\colon \overline{\mathbb{B}}_n &\to \Psi_n \\ \prod_{i<j} e_{ij}^{(k_{ij})} &\mapsto \sum_{i<j} k_{ij}(v_i - v_j). \end{aligned}$$

This makes $\mathfrak{A}_n(\mathbb{K})$ into a Ψ_n-graded algebra. We define subset $\overline{\mathbb{B}}_{n,k}$ of $\overline{\mathbb{B}}_n$ by

$$\overline{\mathbb{B}}_{n,k} := \left\{ \prod_{i<k} e_{ik}^{(l_i)} \middle| l \in \mathbb{N}_0 \right\}.$$

Remark 2.17. Let $\mathfrak{A}_{n,k}(\mathbb{K})$ be the $\mathbb{K}$-vector subspace of $\mathfrak{A}_n(\mathbb{K})$ generated by $\overline{\mathbb{B}}_{n,k}$. Then

$$\mathfrak{A}_n(\mathbb{K}) \cong \mathfrak{A}_{n,n}(\mathbb{K}) \otimes_{\mathbb{K}} \mathfrak{A}_{n,n-1}(\mathbb{K}) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathfrak{A}_{n,2}(\mathbb{K})$$

as $\mathbb{K}$-vector spaces, since

$$\overline{\mathbb{B}}_n = \overline{\mathbb{B}}_{n,n} \times \overline{\mathbb{B}}_{n,n-1} \times \cdots \times \overline{\mathbb{B}}_{n,2}.$$

Note that each $\mathfrak{A}_{n,k}(\mathbb{K})$ is graded over $\Psi_{n,k}$, since $\deg(\overline{\mathbb{B}}_{n,k}) \subset \Psi_{n,k}$. For $l > k$ denote by $\mathfrak{A}_{n,l,k}(\mathbb{K})$ the tensor product

$$\mathfrak{A}_{n,l}(\mathbb{K}) \otimes_{\mathbb{K}} \mathfrak{A}_{n,l-1}(\mathbb{K}) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathfrak{A}_{n,k}(\mathbb{K}).$$

It is a subspace of $\mathfrak{A}_n(\mathbb{K})$ with the basis

$$\overline{\mathbb{B}}_{n,l} \times \overline{\mathbb{B}}_{n,l-1} \times \cdots \times \overline{\mathbb{B}}_{n,k}.$$

Proposition 2.18. *For all $l > k$, the space $\mathfrak{A}_{n,l,k}(\mathbb{K})$ is a subalgebra of $\mathfrak{A}_n(\mathbb{K})$.*

Proof. Let $\mathfrak{g}$ be a Lie subalgebra of sl_n^+. Then, from Poincare-Birkhoff-Witt Theorem, it follows, that $\mathfrak{U}(\mathfrak{g})$ is a subalgebra of $\mathfrak{A}_n(\mathbb{C})$. Note, that

$$\mathfrak{g}_{l,k} = \mathbb{C}\left\langle e_{ij} | i < j,\, k \le j \le l \right\rangle$$

is a Lie subalgebra of sl_n^+, since

$$[e_{ij}, e_{i'j'}] = \begin{cases} e_{ij'} & j = i' \\ -e_{i'j} & i = j' \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathfrak{U}(\mathfrak{g}_{l,k})$ is a subalgebra of $\mathfrak{A}_n(\mathbb{C})$. But now, it follows from the definition that

$$\mathfrak{A}_{n,l,k}(\mathbb{Z}) = \mathfrak{U}(\mathfrak{g}_{l,k}) \cap \mathfrak{A}_n(\mathbb{Z})$$

and, therefore, $\mathfrak{A}_{n,l,k}(\mathbb{Z})$ is a subalgebra of $\mathfrak{A}_n(\mathbb{Z})$. Since the property "to be subalgebra" "commutes" with the functor $\mathbb{K} \otimes_{\mathbb{Z}} -$, it follows, that $\mathfrak{A}_{n,l,k}(\mathbb{K})$ is a subalgebra of $\mathfrak{A}_n(\mathbb{K})$. ◻

2.4 Main results

Define B_n to be the algebra of all $\mathbb{K}$-valued functions on $\mathbb{Z}^n$. Then Ψ_n acts on B_n by shifts

$$f^\gamma(z) := f(z - \gamma).$$

We consider $\Lambda(n,r)$ as a subset of $\mathbb{Z}^n$. Denote by χ the indicator function of $\Lambda(n,r)$, that is

$$\chi(\gamma) = \begin{cases} 1, & \gamma \in \Lambda(n,r) \\ 0, & \gamma \notin \Lambda(n,r). \end{cases}$$

Then χ is an idempotent in the algebra B, and therefore $e = 1_{\mathfrak{A}_n(\mathbb{K})} \otimes \chi$ is an idempotent in the algebra $\mathfrak{C}_n(\mathbb{K}) := \mathfrak{A}_n(\mathbb{K}) \ltimes_{\Psi_n} B$. Denote by $\bar{e}$ the idempotent $1 - e$ in $\mathfrak{A}_n(\mathbb{K}) \ltimes_\Gamma B$.

Theorem 2.19. *The ideal $\mathfrak{C}_n(\mathbb{K})\bar{e}\,\mathfrak{C}_n(\mathbb{K})$ of $\mathfrak{C}_n(\mathbb{K})$ is strong idempotent and the set*

$$\mathbb{J} = \left\{ \prod_{i<j} e_{ij}^{(k_{ij})} \chi_\mu \,\middle|\, \lambda, \mu \in \Lambda(n,r),\ \lambda \trianglerighteq \mu,\ (k_{ij})_{i,j=1}^n \in T(\lambda,\mu) \right\}$$

is a basis of

$$\mathfrak{C}_n(\Lambda(n,r)) = \mathfrak{C}_n(\mathbb{K}) \big/ \mathfrak{C}_n(\mathbb{K})\bar{e}\,\mathfrak{C}_n(\mathbb{K})$$

Proof. We will apply Theorem 1.35 to the situation

(i) $A = \mathfrak{A}_n(\mathbb{K})$;

(ii) $m = n-1$;

(iii) $A_1 = \mathfrak{A}_{n,n}(\mathbb{K}),\ A_2 = \mathfrak{A}_{n,n-1}(\mathbb{K}),\ \ldots,\ A_{n-1} = \mathfrak{A}_{n,1(\mathbb{K})}$;

(iv) $J_1 = \overline{\mathbb{B}}_{n,n},\ J_2 = \overline{\mathbb{B}}_{n,n-1},\ \ldots,\ J_{n-1} = \overline{\mathbb{B}}_{n,1}$;

(v) $\Gamma_1 = \Psi_{n,n},\ \Gamma_2 = \Psi_{n,n-1},\ \ldots,\ \Gamma_{n-1} = \Psi_{n,1}$;

(vi) $Y = \Lambda^1(n,r)$;

(vii) $Z_1 = \Lambda(n,r),\ Z_2 = M^{n-1}(n,r),\ \ldots,\ Z_{n-1} = M^2(n,r)$ with the orderings $\leq_k$ considered in Proposition 2.6 on them.

Let us check that the conditions of Theorem 1.35 are satisfied:

(i) The monoid Ψ_n is commutative.

(ii) By Proposition 2.18, the subspaces

$$A_{ij} = A_i \otimes A_{i+1} \otimes \cdots \otimes A_j = \mathfrak{A}_{n,l,k}(\mathbb{K}) \text{ for } l = n+1-i \text{ and } k = n+1-j$$

are subalgebras of $A = \mathfrak{A}_n(\mathbb{K})$ for all $1 \leq i,j \leq n$.

(iii) The algebras $\mathfrak{A}_{n,k}(\mathbb{K})$ are $\Psi_{n,k}$-graded, since $\deg(\overline{\mathbb{B}}_{n,k}) \subset \Psi_{n,k}$.

(iv) Let $j \geq i$ and $l = n+1-j$ and $k = n+1-i$. Then

$$\begin{aligned}
\Gamma_j Z_i \cap Y &= \Psi_{n,l} M^k(n,r) \cap \Lambda^1(n,r) \\
&= \left\{ z' \,\middle|\, z' = z + \sum_{\tau=1}^{l-1} k_\tau (v_\tau - v_l);\ \begin{array}{l} z_1 \geq 0, \ldots, z_{k-1} \geq 0, z_k < 0; \\ z, z' \in \Lambda^1(n,r); k_\tau \in \mathbb{N}_0 \end{array} \right\}.
\end{aligned}$$

If $l = k$, then the last set coincides with $M^k(n,r)$ and therefore

$$\Gamma_j Z_j \cap Y = M^k(n,r) = Z_j.$$

If $l < k$, then

$$\left\{ z' \,\middle|\, z' = z + \sum_{\tau=1}^{l-1} k_\tau(v_\tau - v_l);\ \begin{array}{l} z_1 \geq 0, \ldots, z_{k-1} \geq 0, z_k < 0; \\ z, z' \in \Lambda^1(n,r); k_\tau \in \mathbb{N}_0 \end{array} \right\} =$$

$$= \left\{ z' \,\middle|\, \begin{array}{l} z'_1 \geq 0, \ldots, z'_{l-1} \geq 0, z'_l < 0; \\ z'_{l+1} \geq 0, \ldots, z'_{k-1} \geq 0, z'_k < 0 \\ z' \in \Lambda^1(n,r) \end{array} \right\}$$

$$\bigsqcup \left\{ z' \,\middle|\, \begin{array}{l} z'_1 \geq 0, \ldots, z'_{l-1} \geq 0, z'_l \geq 0; \\ z'_{l+1} \geq 0, \ldots, z'_{k-1} \geq 0, z'_k < 0 \\ z' \in \Lambda^1(n,r) \end{array} \right\}$$

$$\subset M^l(n,r) \sqcup M^k(n,r) = Z_i \sqcup Z_j \subset \coprod_{\tau=i}^{j} Z_\tau.$$

(v) Let $l = n+1-j$. Then $Y_j = Z_1 \sqcup Z_2 \sqcup \cdots \sqcup Z_j = \Lambda^l(n,r)$. Hence for $i < j$ and $k = n+1-i$

$$\begin{aligned} \Gamma_i Y_j \cap Y &= \Psi_{n,k}\Lambda^l(n,r) \cap \Lambda^1(n,r) \\ &= \left\{ z' \,\middle|\, z' = z + \sum_{\tau=1}^{k-1} k_\tau(v_\tau - v_k);\ \begin{array}{l} z_1 \leq 0, z_2 \leq 0, \ldots, z_l \leq 0; \\ z, z' \in \Lambda^1(n,r); k_\tau \in \mathbb{N}_0 \end{array} \right\} \\ &= \Lambda^l(n,r) \end{aligned}$$

since $k > l$, and therefore the first l coordinates of z', which are obtained from the first l coordinates of z upon addition of $\sum_{\tau=1}^{k-1} k_\tau(v_\tau - v_k)$, are positive.

(vi) The radical of $A_0 = \mathbb{K}$ is zero.

(vii) The orders $\leq_k$ on $M^k(n,r)$ satisfy the additional conditions of Proposition 2.6.

Therefore an ideal $\mathfrak{C}_n(\mathbb{K})\bar{e}\,\mathfrak{C}_n(\mathbb{K})$ is strong idempotent, and the set

$$\mathbb{I} = \left\{ a_n a_{n-1} \ldots a_2 \chi_\mu \,\middle|\, \begin{array}{l} a_k \in \overline{\mathbb{B}}_{n,k},\ 2 \leq k \leq n-1; \mu \in \Lambda(n,r) \\ \mu + \deg(a_k a_{k-1} \ldots a_2) \in \Lambda(n,r),\ 2 \leq k \leq n \end{array} \right\}.$$

is a basis of $\mathfrak{C}_n(\mathbb{K})(\Lambda(n,r))$. Let $a_n a_{n-1} \ldots a_2 \chi_\mu$ be an element of $\mathbb{I}$. Since $\deg(a_j) \in \Psi_{n,j}$, there are $k_{ij} \in \mathbb{N}_0$, such that

$$\deg(a_j) = \sum_{i=1}^{j-1} k_{ij}(v_i - v_j).$$

Let

$$k_{jj} = \mu_j - \sum_{i=1}^{j-1} k_{ij}$$

for $j = 1, 2, \dots, n$. Note that k_{jj} is the j-th coordinate of $z_j = \mu + \deg(a_j) + \deg(a_{j-1}) + \cdots + \deg(a_2)$. Since $z_j \in \Lambda(n, r)$ it follows, that $k_{jj} \in \mathbb{N}_0$ for all j. Define $\lambda_i = \sum_{j=i}^{n} k_{ij}$. Then $(k_{ij})_{i,j=1}^n \in T(\lambda, \mu)$. Thus $\mathbb{I} \subset \mathbb{J}$.

Now, let $\prod_{i<j} e_{ij}^{(k_{ij})} \chi_\mu$ be an element of $\mathbb{J}$. Then $(k_{ij})_{i,j=1}^n$ is an element of $T(\lambda, \mu)$ for some $\lambda \trianglerighteq \mu$. We set

$$a_j = \prod_{i=1}^{j-1} e_{ij}^{(k_{ij})}$$

for $j \in \{2, 3, \dots, n\}$. Then a_j is an element of $\overline{\mathbb{B}}_{n,j}$. Now, for all $k < j$, we have that the j-th coordinate of $\mu + \deg(a_k) + \cdots + \deg(a_2)$ is the same as in μ, and thus it is positive. For $k = j$, this j-th coordinate is equal to

$$\mu_j - \sum_{i=1}^{j-1} k_{ij} = \sum_{i=1}^{j} k_{ij} - \sum_{i=1}^{j-1} k_{ij} = k_{jj} \geq 0.$$

For $k \geq j$ the j-th coordinate of $\mu + \deg(a_k) + \cdots + \deg(a_2)$ is

$$\mu_j - \sum_{i=1}^{j-1} k_{ij} + \sum_{i=j+1}^{k} k_{ji} = \sum_{i=j}^{k} k_{ji} \geq 0.$$

Therefore, for every k, the element $\mu + \deg(a_k) + \cdots + \deg(a_2)$ lies in $\Lambda(n, r)$. Therefore $\mathbb{J} \subset \mathbb{I}$. $\square$

Theorem 2.20. *The algebra $\mathfrak{C}_n(\Lambda(n, r))$ is isomorphic to the Borel subalgebra $S^+(n, r)$ of the Schur algebra $S(n, r)$.*

Before we prove the theorem let us explain how we can use it to construct minimal projective resolutions of the simple $S^+(n, r)$-modules.

Suppose, we have constructed a Ψ_n-graded (minimal) projective resolution $P_\bullet$ of the trivial module M over the algebra $\mathfrak{A}_n(\mathbb{K})$. For each $\lambda \in \Lambda(n, r)$ we denote by N_λ the one dimensional B_n-module, such that $fv = f(\lambda)v$ for all $v \in N_\lambda$ and all $f \in B_n$. Then, by Proposition 1.17 and Corollary 1.19, $P_\bullet \ltimes_{\Psi_n} N_\lambda$ is a Ψ_n-graded (minimal) projective resolution of the module $M \ltimes_{\Psi_n} N_\lambda$, since N_λ is a projective B_n-module, $\mathfrak{A}_n(\mathbb{K})_0 = \mathbb{K}$ and $\mathrm{Rad}(B_n) = 0$. Now, the complex

$$\mathfrak{C}_n(\Lambda(n, r)) \otimes_{\mathfrak{C}_n(\mathbb{K})} (P_\bullet \ltimes_{\Psi_n} N_\lambda)$$

is a (minimal) Ψ_n-*graded* projective resolution of the module

$$\mathfrak{C}_n(\Lambda(n, r)) \otimes_{\mathfrak{C}_n(\mathbb{K})} (M \ltimes_{\Psi_n} N_\lambda) \cong \mathbb{K}_\lambda.$$

Since $\mathfrak{C}_n(\Lambda(n, r)) \cong S^+(n, r)$ is a finite dimensional algebra, by Proposition 1.13, the complex

$$\mathfrak{C}_n(\Lambda(n, r)) \otimes_{\mathfrak{C}_n(\mathbb{K})} (P_\bullet \ltimes_{\Psi_n} N_\lambda)$$

is a (minimal) projective resolution of the $S^+(n, r)$-module $\mathbb{K}_\lambda$ in the ungraded sense.

Proof. In this proof we will use the notation introduced in the beginning of Section 2.2. For simplicity we will write u for the element $u \otimes 1_B$ of $\mathfrak{C}_n(\mathbb{K})$.

Let $\{e_k \mid 1 \leq k \leq n\}$ be a basis of V. Denote by E_{ij} an endomorphism of V given by $E_{ij}(e_k) = \delta_{jk} e_i$. Define a representation ρ_r of $\mathfrak{A}_n(\mathbb{K})$ on $V^{\otimes r}$ by

$$\rho_r\left(e_{ij}^{(k)}\right)(v_1 \otimes \cdots \otimes v_r) =$$
$$= \sum_{\sigma_1 < \sigma_2 < \cdots < \sigma_k} v_1 \otimes \cdots \otimes E_{ij}(v_{\sigma_1}) \otimes \cdots \otimes E_{ij}(v_{\sigma_2}) \otimes \cdots \otimes E_{ij}(v_{\sigma_k}) \otimes \cdots \otimes v_r.$$

For $\lambda \in \Lambda(n,r)$, write ξ_λ for $\xi_{i,i}$, for any $i \in I(n,r)$ such that $\mathrm{wt}(i) = \lambda$. Extend ρ_r to $\mathfrak{C}_n(\mathbb{K})$ by

$$\rho_r(a \otimes \chi_\lambda) = \begin{cases} \rho_r(a)\xi_\lambda & \text{if } \lambda \in \Lambda(n,r) \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\rho_r\left(\chi_{\overline{\Lambda(n,r)}}\right) = 0$. Therefore ρ_r is a representation of the algebra $\mathfrak{C}_n(\Lambda(n,r))$. Note, that the image $\mathrm{Im}(\rho_r)$ of ρ_r is a subalgebra of $\mathrm{End}(V^{\otimes r})$.

First we show that the image $\mathrm{Im}(\tau_r)$ of $\tau_r = \tau_{n,r}$ is a subset of the image $\mathrm{Im}(\rho_r)$. The group $B_n^+(K)$ is generated by the elements of the form $I + \mu E_{ij}$, where I is the identity matrix, $\mu \in \mathbb{K}$ and $i \leq j$. It is easy to check, that

$$\tau_r(I + \mu E_{ii}) = \sum_{\lambda \in \Lambda(n,r)} (1+\mu)^{\lambda_i} \xi_\lambda = \sum_{\lambda \in \Lambda(n,r)} (1+\mu)^{\lambda_i} \rho_r(\chi_\lambda).$$

Suppose $i < j$. Then

$$\begin{aligned}\tau_r(I + \mu E_{ij})(v_1 \otimes \cdots \otimes v_r) &= (v_1 + \mu E_{ij} v_1) \otimes \cdots \otimes (v_r + \mu E_{ij} v_r) \\ &= \sum_{k=0}^{r} \mu^k \rho_r\left(E_{ij}^{(k)}\right)(v_1 \otimes \cdots \otimes v_r).\end{aligned}$$

This shows, that $\mathrm{Im}(\tau_r) \subset \mathrm{Im}(\rho_r)$.

Thus $S^+(n,r)$ is a subalgebra of $\mathrm{Im}(\rho_r)$, and therefore it is a subquotient of $\mathfrak{C}_n(\Lambda(n,r))$. Now we use the fact that both algebras have bases that are in bijection with the finite set $T(n,r)$ (see Proposition 2.14 and Theorem 2.19). Therefore, they have equal dimensions and are isomorphic.

□

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